Geodesic-Loxodromes for Diffusion Tensor Interpolation and Difference Measurement

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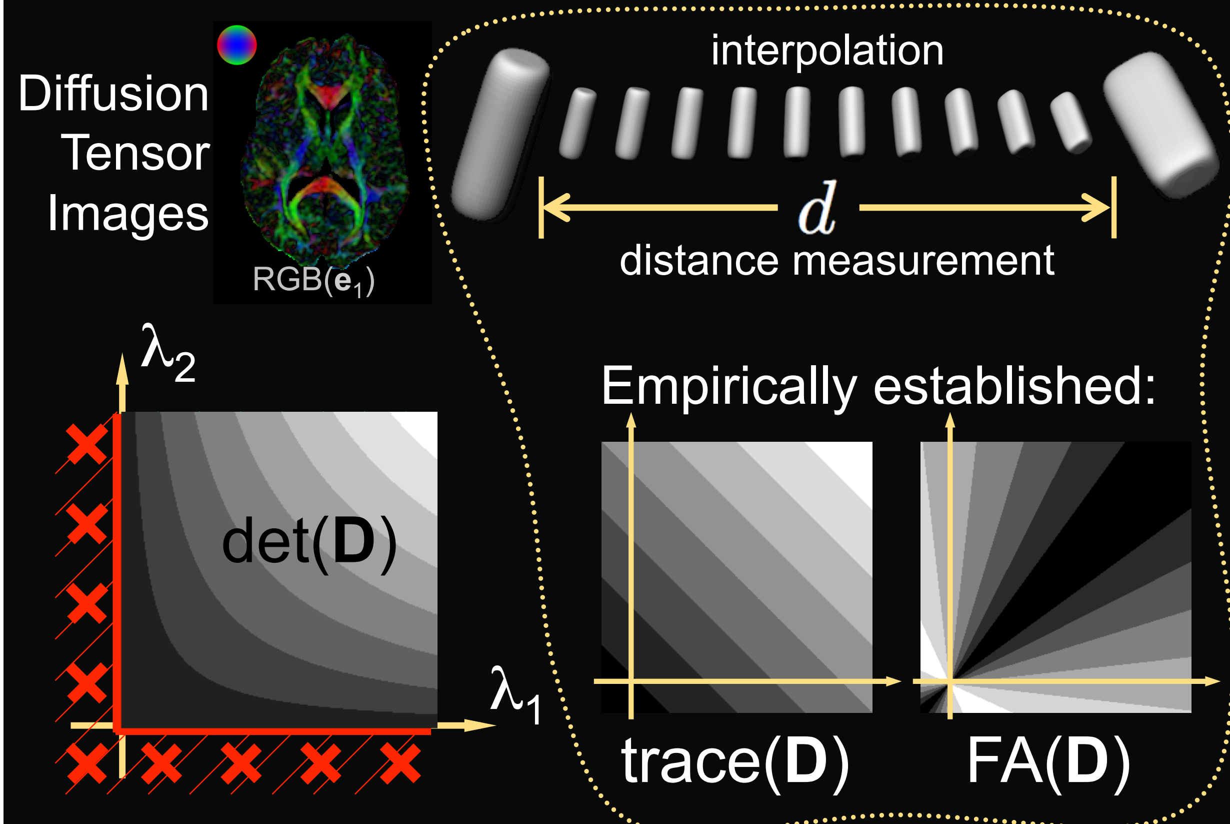
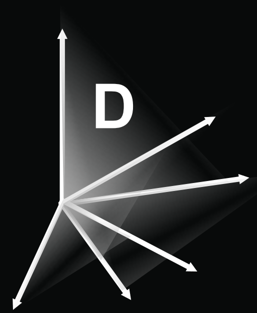
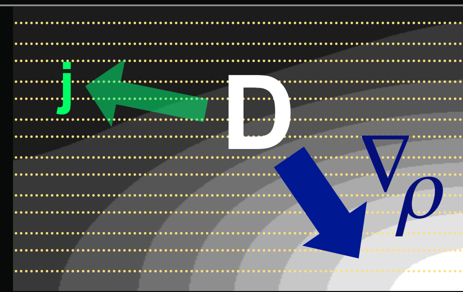
Introduction



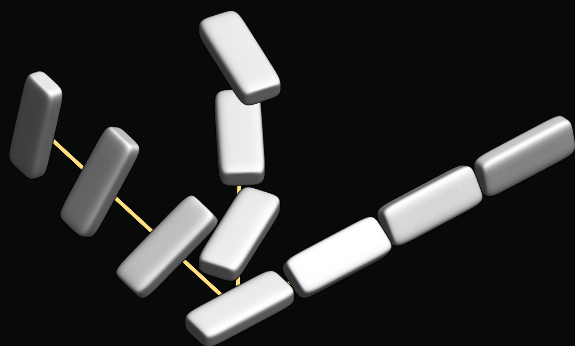
Introduction

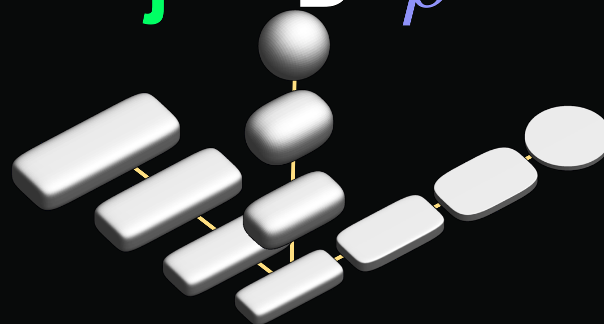


Diffusion tensors as elements of vector space



$$\mathbf{j} = -\mathbf{D}\nabla\rho$$





Orientation: Geodesic

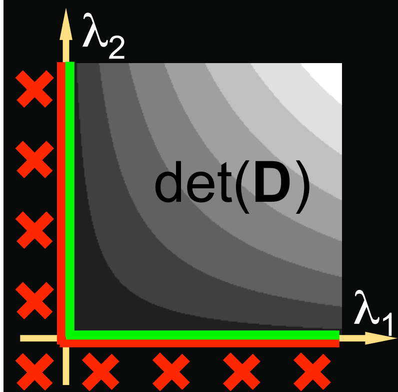
Shape: Loxodrome

Distance measures from path integrals

Rest of talk

- Previous work
- Tensor shape
- Loxodromes
- Geodesic-Loxodromes, Results
- Distance measures, Results
- Conclusions

Riemannian, Log-Euclidean approaches



Riemannian: Batchelor et al. 2005, Pennec et al. 2006, Fletcher et al. 2007, Lenglet et al. 2007

$$F(t,\mathbf{A},\mathbf{B}) = \mathbf{A}^{1/2}\,(\mathbf{A}^{-1/2}\,\mathbf{B}\,\mathbf{A}^{-1/2})^{t}\,\mathbf{A}^{1/2}$$

Log-Euclidean: Arsigny et al. 2006

$$F(t,\mathbf{A},\mathbf{B}) = \exp((1-t)\log(\mathbf{A}) + t\log(\mathbf{B}))$$

Both monotonically interpolate det(**D**); no "swelling":

Linear:

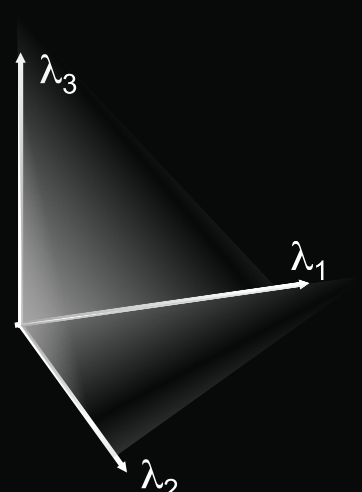
Riemannian:

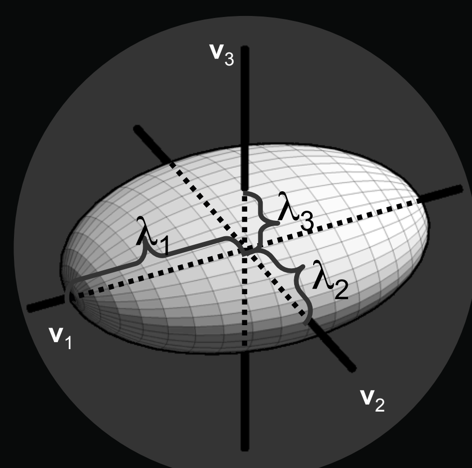
Determinant is only one possible size measure…

Space of Tensor Shape

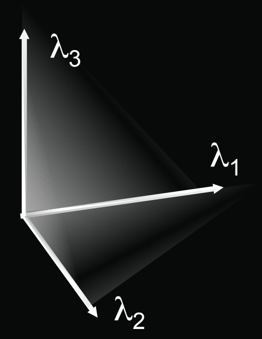
$$\mathbf{D} = \mathbf{R}\Lambda\mathbf{R}^{-1}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix}$$





Parameterizing Shape with Invariants



tr(**D**) = Dxx+Dyy+Dzz

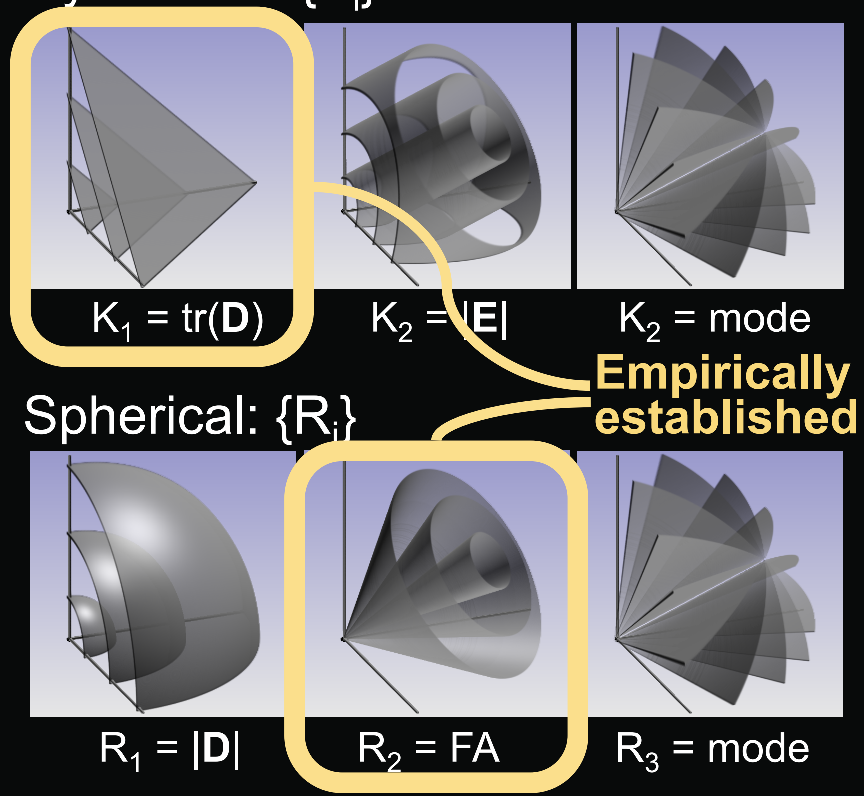
|**D**| = sqrt(tr($\mathbf{D}^T\mathbf{D}$))

E = deviatoric(**D**)
= **D** - tr(**D**)*I/3

mode = det(**E**/|**E**|)

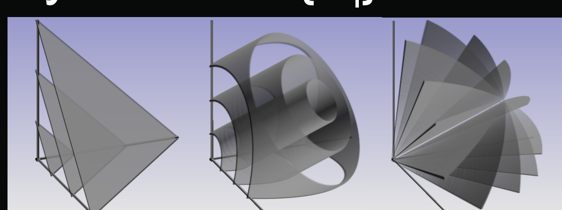
Cylindrical: $\{K_i\}$

Spherical: $\{R_i\}$



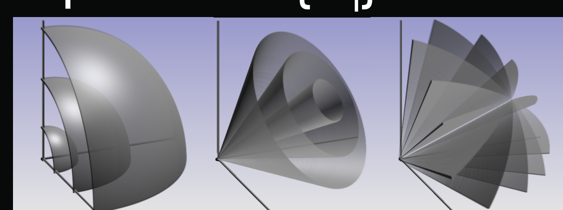
Orthogonal invariant sets

Cylindrical: $\{K_i\}$



$$\nabla K_i : \nabla K_j = 0 \text{ if } i \neq j$$

Spherical: $\{R_i\}$



$$\nabla R_i : \nabla R_j = 0 \text{ if } i \neq j$$

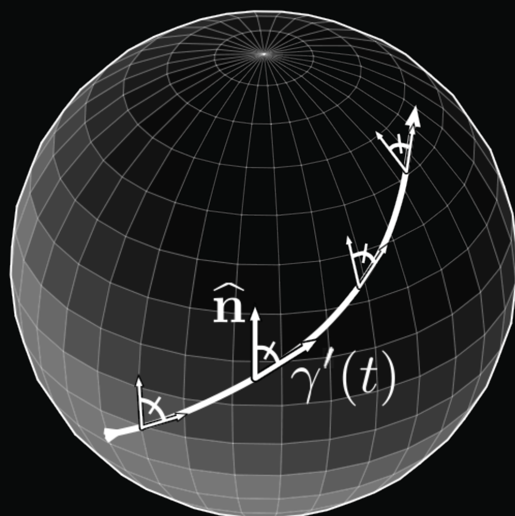
- $\{K_i\}$ and $\{R_i\}$ are **orthogonal** sets of shape parameters: their gradients are mutually orthogonal (Ennis & Kindlmann MRM 2006)
- {det(**D**),**?**,**?**} third invariant set?

$\Rightarrow$ Geodesic-Loxodromes monotonically interpolate $\{K_i\}$ or $\{R_i\}$

Loxodrome defined

Path $\gamma(t)$ of constant speed and bearing ⇔
Fixed angle with respect to North $\widehat{\mathbf{n}}(\mathbf{x})$

$$|\gamma'(t)| = 1 \text{ and } \gamma'(t) \cdot \widehat{\mathbf{n}}(\gamma(t)) = \alpha \text{ for all } t$$



Loxodromes monotonically interpolate **latitude**

Geodesic-Loxodrome defined, intuitively

Minimal-length path between two tensors that monotonically interpolates tensor shape

Geodesic-Loxodrome defined, mathematically

Geodesic-loxodrome $\gamma(t)$ between **A** and **B** is the *shortest* path satisfying:

$$\boldsymbol{\gamma}(0) = \mathbf{A}, \;\; \boldsymbol{\gamma}(l) = \mathbf{B}, \;\; |\boldsymbol{\gamma}'(t)| = 1, \;\; \text{and}$$
$$\boldsymbol{\gamma}'(t) \mathbin{:} \widehat{\boldsymbol{\nabla}} J_i(\boldsymbol{\gamma}(t)) = \alpha_i \text{ for all } t \in [0, l], i \in \{1, 2, 3\}$$

where $J_i = K_i$ or $J_i = R_i$ and $\widehat{\boldsymbol{\nabla}} J_i = \dfrac{\boldsymbol{\nabla} J_i}{|\boldsymbol{\nabla} J_i|}$

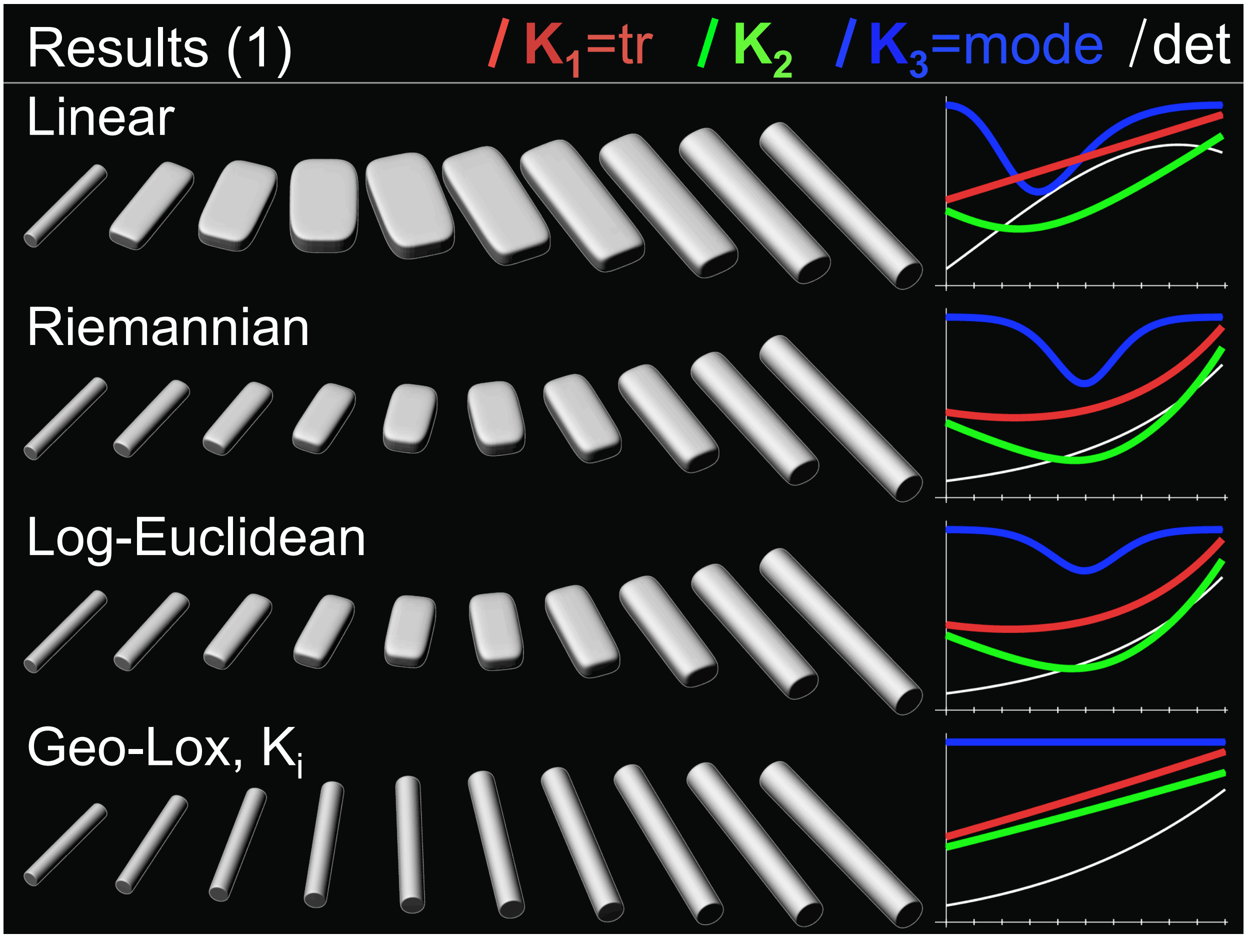
Constants α_i and l are path “bearing” and length

Monotonic: $\dfrac{d}{dt} J_i(\boldsymbol{\gamma}(t)) = \boldsymbol{\gamma}'(t) \mathbin{:} \boldsymbol{\nabla} J_i(\boldsymbol{\gamma}(t)) = \alpha_i |\boldsymbol{\nabla} J_i(\boldsymbol{\gamma}(t))|.$

$$J_i(\mathbf{A}) = J_i(\mathbf{B}) \Rightarrow J_i(\mathbf{A}) = J_i(\boldsymbol{\gamma}(t)) = J_i(\mathbf{B}) \text{ for all } t$$

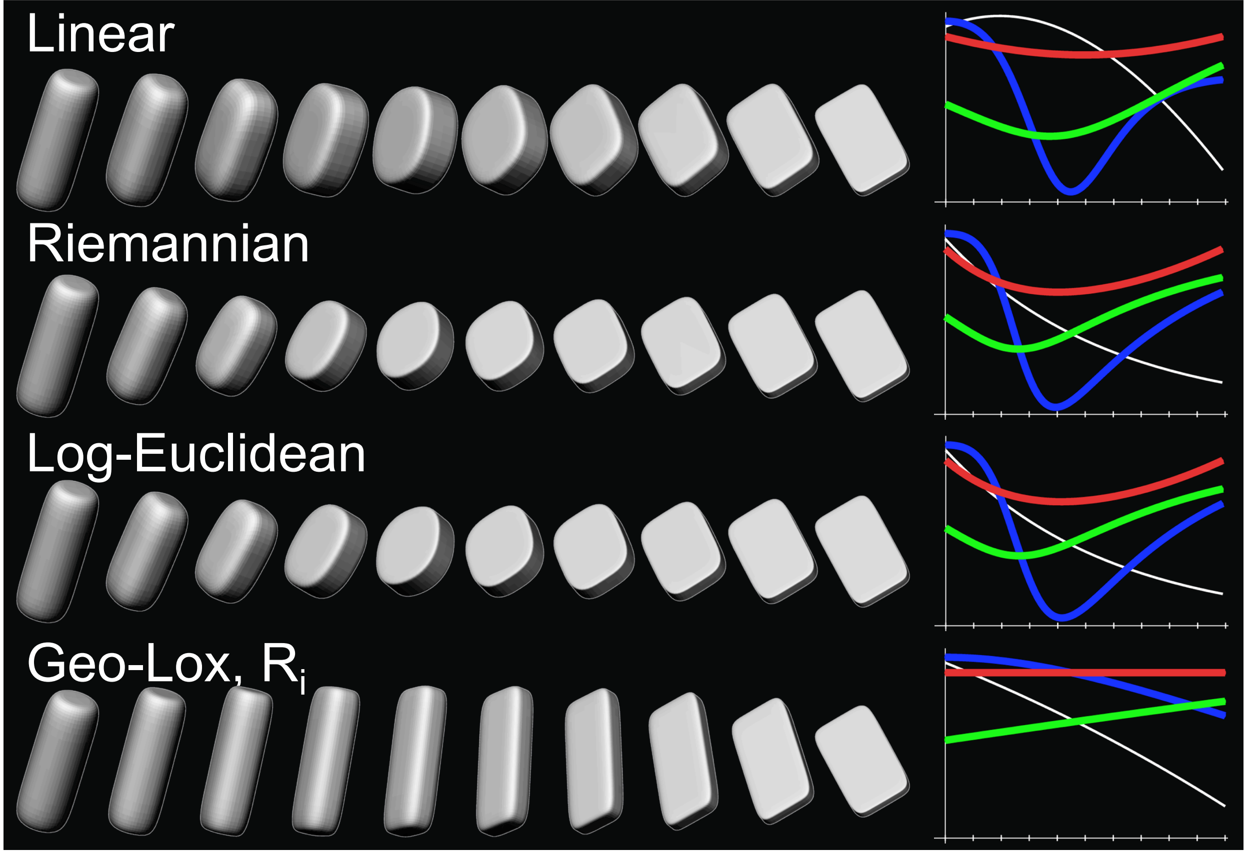
Intersection of isocontours = rotation orbit; geodesic

Results (1)



Results (2)

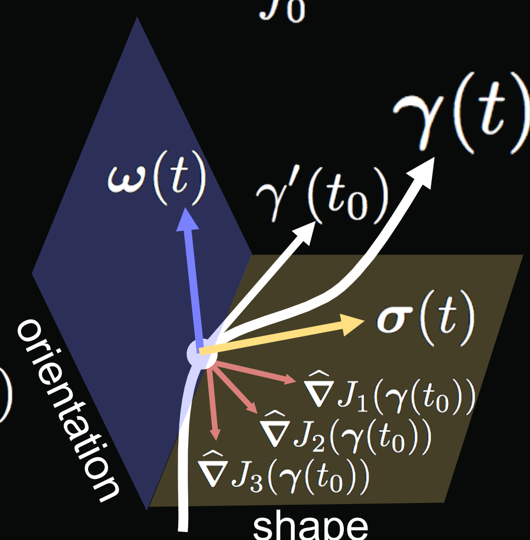
/R_1=|D| /R_2=FA /R_3=mode /det



Distance measurement

Integrate tangent norm: $d(\mathbf{A},\mathbf{B}) = \int_0^l |\gamma'(t)|dt = l$

Or, exploit **shape** and **orientation** subspaces:



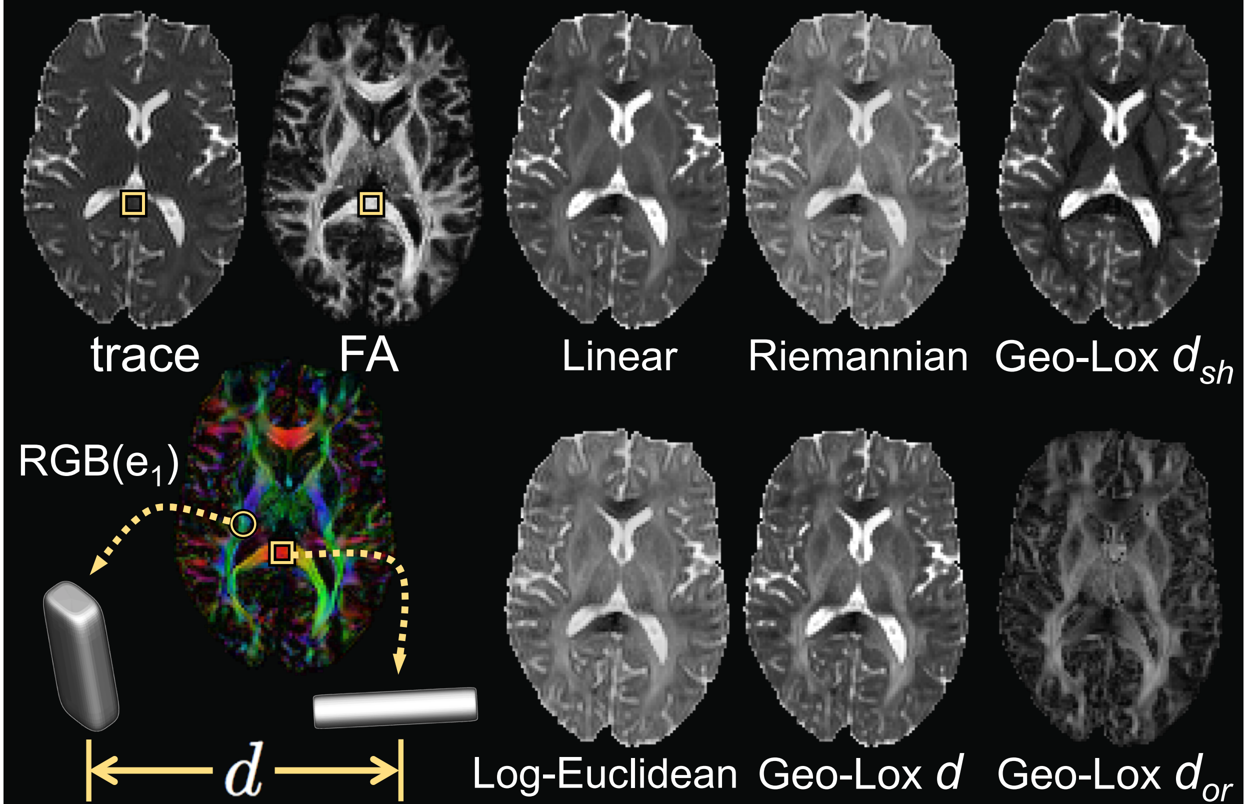
$$\boldsymbol{\omega}(t) = \boldsymbol{\gamma}'(t) - \boldsymbol{\sigma}(t)$$

$$d_{or}(\mathbf{A},\mathbf{B}) = \int_0^l |\boldsymbol{\omega}(t)|dt$$

$$\boldsymbol{\sigma}(t) = \sum_i \boldsymbol{\gamma}'(t) : \hat{\boldsymbol{\nabla}} J_i(\boldsymbol{\gamma}(t)) \hat{\boldsymbol{\nabla}} J_i(\boldsymbol{\gamma}(t))$$

$$d_{sh}(\mathbf{A},\mathbf{B}) = \int_0^l |\boldsymbol{\sigma}(t)|dt$$

Distance measurement, Results



Conclusions, Future Work

- Intuition about orientation and shape mapped to formulation of geodesic and loxodrome properties
- Structure math of interpolation around established tensor **parameters**, not just positive-definiteness
- Tensor Mode ($K_3 = R_3$) important for shape
- Working on fast approximations
- Need log(), exp() map analogs for multi-linear

Acknowledgements

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Thank you