Superquadric Glyphs for Symmetric Second-Order Tensors

Thomas Schultz, Gordon L. Kindlmann

Computer Science Dept, Computation Institute

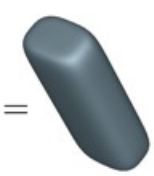
University of Chicago

THE UNIVERSITY OF CHICAGO COMPUTER SCIENCE

Computation Institute

Symmetric Tensor Representations

$$\mathbf{D} = \begin{pmatrix} 3.08 & 1.21 & 0.77 \\ 1.21 & 3.85 & 1.98 \\ 0.77 & 1.98 & 5.06 \end{pmatrix}$$

$\mathbf{D} =$ 

[Kindlmann 2004]

$$\mathbf{D} = \underbrace{\begin{pmatrix} -0.33 & -0.74 & 0.59 \\ -0.59 & -0.33 & -0.74 \\ -0.74 & 0.59 & 0.33 \end{pmatrix}}_{R} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} -0.33 & -0.59 & -0.74 \\ -0.74 & -0.33 & 0.59 \\ 0.59 & -0.74 & 0.33 \end{pmatrix}}_{R^{-1}}$$

↑

Eigenvalues tell us about Positive Definiteness

Symmetric Tensor Representations

$$\mathbf{D} = \begin{pmatrix} 2.03 & 2.52 & 0.19 \\ 2.52 & 2.22 & 2.71 \\ 0.19 & 2.71 & 4.74 \end{pmatrix} \qquad \mathbf{D} = ?$$

Topic of this talk

$$\mathbf{D} = \underbrace{\begin{pmatrix} -0.33 & -0.74 & 0.59 \\ -0.59 & -0.33 & -0.74 \\ -0.74 & 0.59 & 0.33 \end{pmatrix}}_{R} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} -0.33 & -0.59 & -0.74 \\ -0.74 & -0.33 & 0.59 \\ 0.59 & -0.74 & 0.33 \end{pmatrix}}_{R^{-1}}$$

↑

Tensor is indefinite

Why care about indefinite tensors?

Indefinite Symmetric Tensors arise in many fields:

- Hessians
- Stress Tensors
- Rate-of-deformation Tensors
- Geometry Tensors

Principles for Tensor Glyph Design

Faithful and expressive visualization requires:

- **Preservation of Symmetry:** Glyph should have same symmetries as the tensor

$$\mathbf{D} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1} \Longleftrightarrow G(\mathbf{D}) = \mathbf{T}\,G(\mathbf{D})$$

- **Continuity:**

$$\mathbf{D}_1 \approx \mathbf{D}_2 \Longleftrightarrow \mathrm{appearance}(G(\mathbf{D}_1)) \approx \mathrm{appearance}(G(\mathbf{D}_2))$$

- **Disambiguity:**

$$\mathbf{D}_1 \neq \mathbf{D}_2 \Longleftrightarrow \mathrm{appearance}(G(\mathbf{D}_1)) \neq \mathrm{appearance}(G(\mathbf{D}_2))$$

Principles for Tensor Glyph Design

Natural for a wide range of applications:

- **Invariance under scaling:**

$$G(\mathbf{D}) = s(\|\mathbf{D}\|)B\left(\frac{\mathbf{D}}{\|\mathbf{D}\|}\right)$$

- **Invariance under projection to eigenplanes:**

$$G(\mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}) = \mathbf{P}\,G(\mathbf{D})$$

Glyph Coloring

We color each point **x** on the glyph by

$$\mathrm{sign}(\mathbf{x}^T \mathbf{D} \mathbf{x})$$

\+ red

\- blue

Satisfies

- ✓ Preservation of Symmetry
- ✓ Continuity
- ✓ Disambiguity

Our Fundamental Glyph Equation

$$G(\mathbf{D}) = s(\|\mathbf{D}\|)\,\mathbf{R}\,\tilde{\Lambda}\,B(\tilde{\lambda}_i)$$

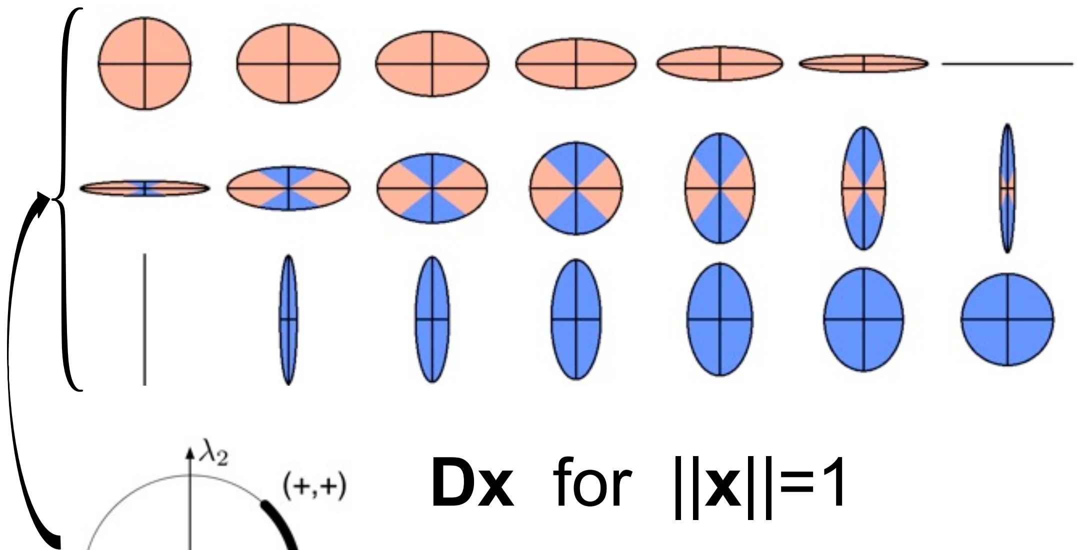
scaling

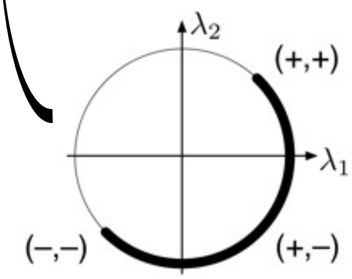
rotation

normalized eigenvalues

basis shape

Explicit Ellipses

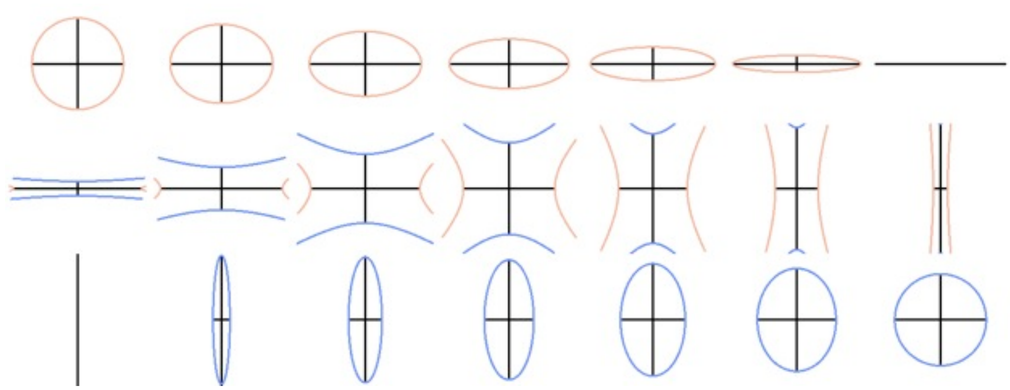




Dx for ||**x**||=1

- Symmetry Preservation
- Disambiguity

Implicit Ellipses

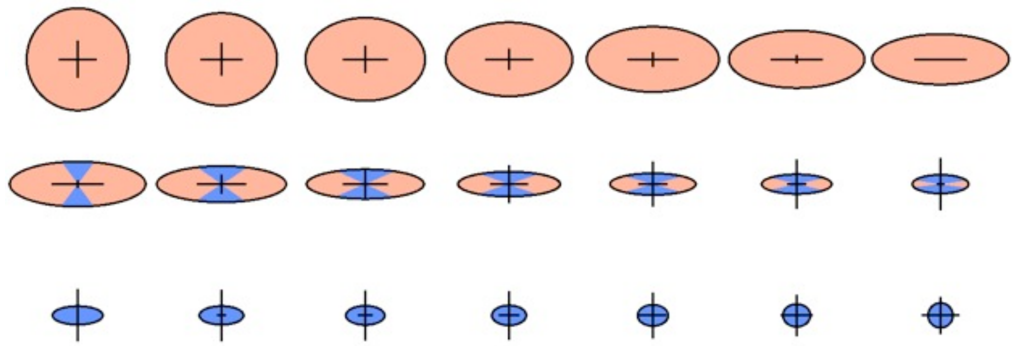


$$\mathbf{x}^T\mathbf{D}^{-2}\mathbf{x} = 1$$

"sign preserving"

- Unbounded Surfaces

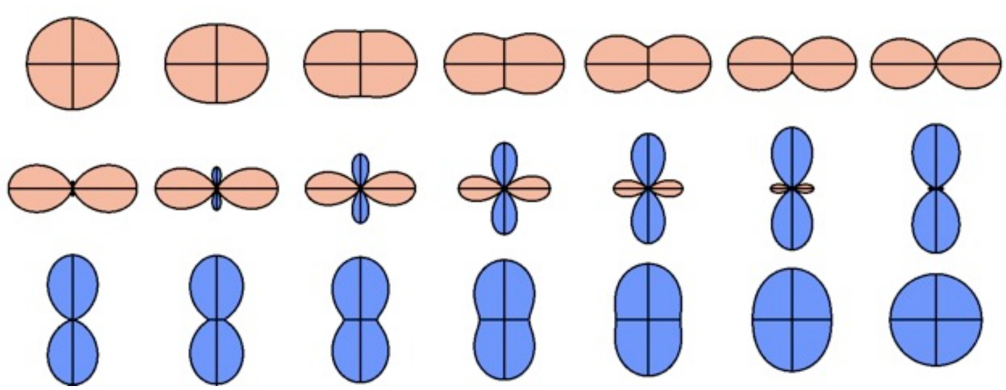
“Exp-Ellipses”



$$\exp(\mathbf{D})\mathbf{x} \quad \text{for} \quad ||\mathbf{x}||=1$$

- ✗ Disambiguity

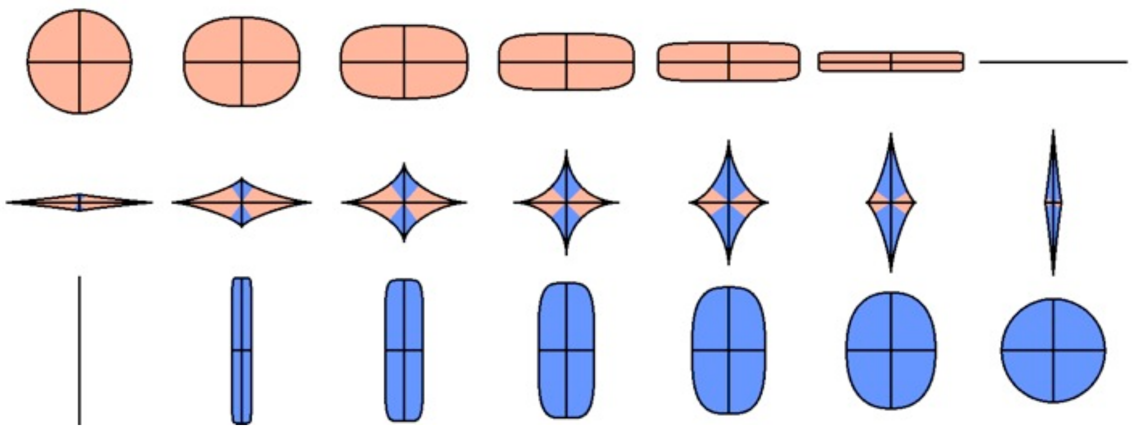
Reynolds Glyphs



$$(\mathbf{x}^T\mathbf{D}^{-2}\mathbf{x})\mathbf{x} \quad \text{for} \quad ||\mathbf{x}||=1$$

- ✗ Invariance under Projection
- 💣 Occlusions in 3D

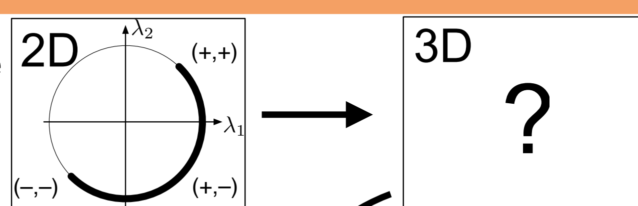
New Glyph for 2D symmetric tensor



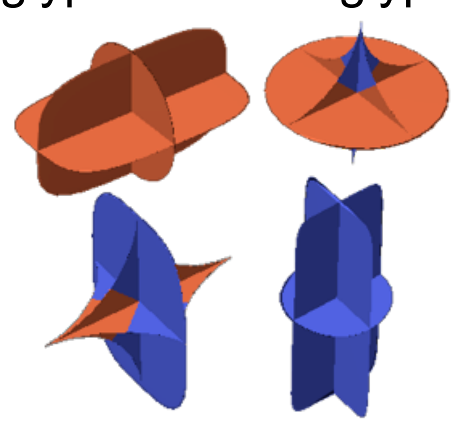
- Meet disambiguity, symmetry preservation req.
- Glyph shape shows eigenvalue sign **differences**
- Convex indicates **same** sign (positive-definite, negative-definite)
- Concave indicates **different** sign (indefinite)

From 2D to 3D tensor glyphs

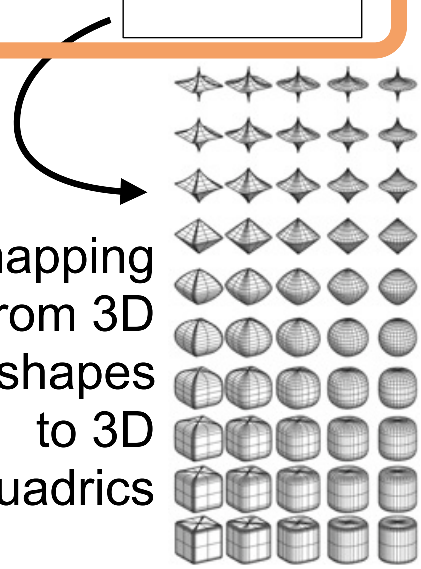
Understand space of 3D symmetric tensor shapes



Scaffold (prototype) 3D glyphs with 2D glyphs

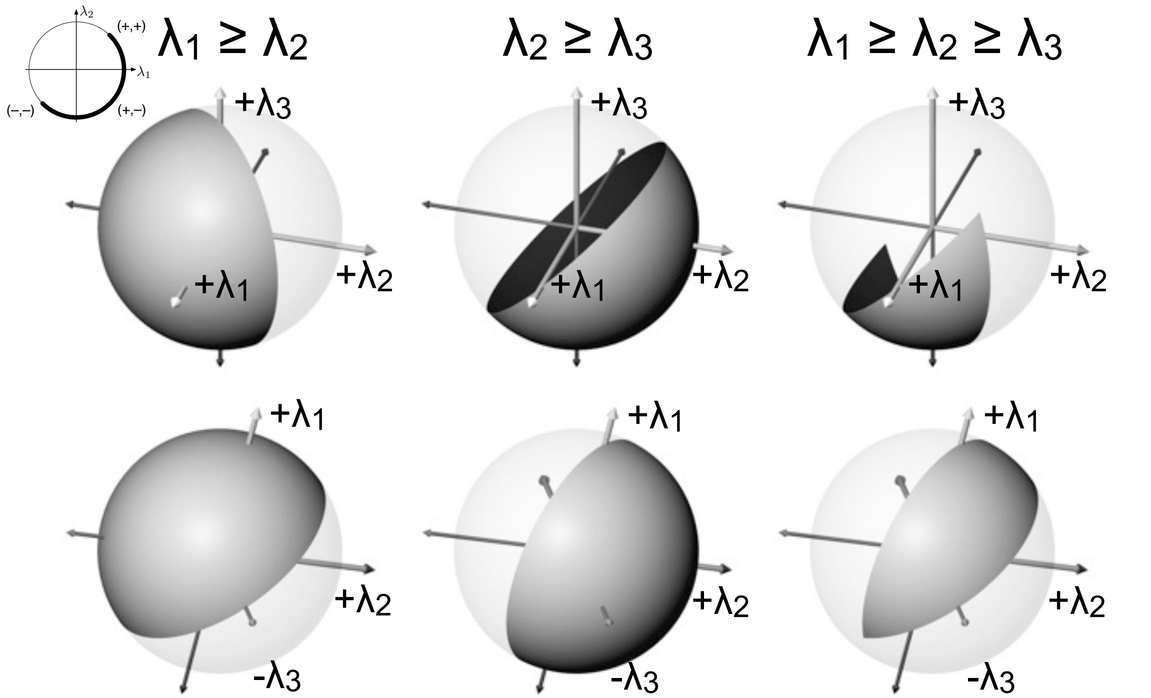


Find mapping from 3D tensor shapes to 3D superquadrics



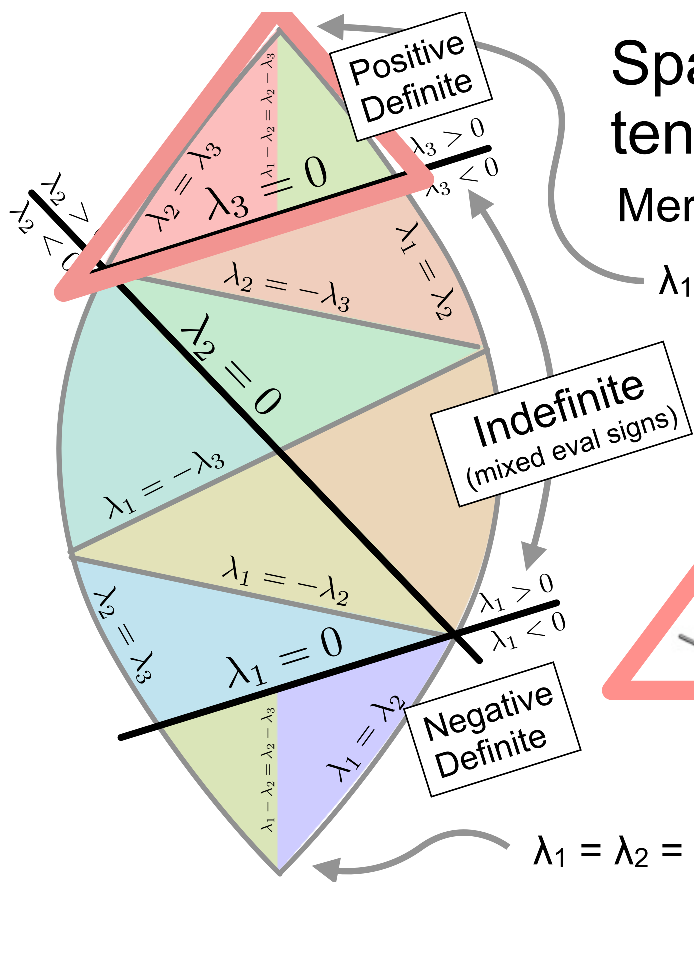
Space of 3D tensor shape

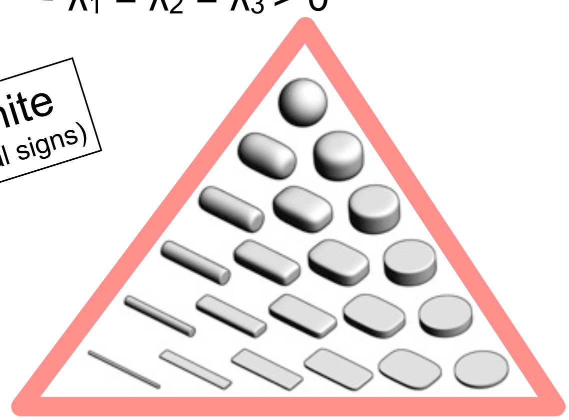
Eigenvalue sorting $\lambda_1 \geq \lambda_2 \geq \lambda_3$ ⇒ **Lune** of tensor shape



Space of 3D tensor shape

Mercator projection

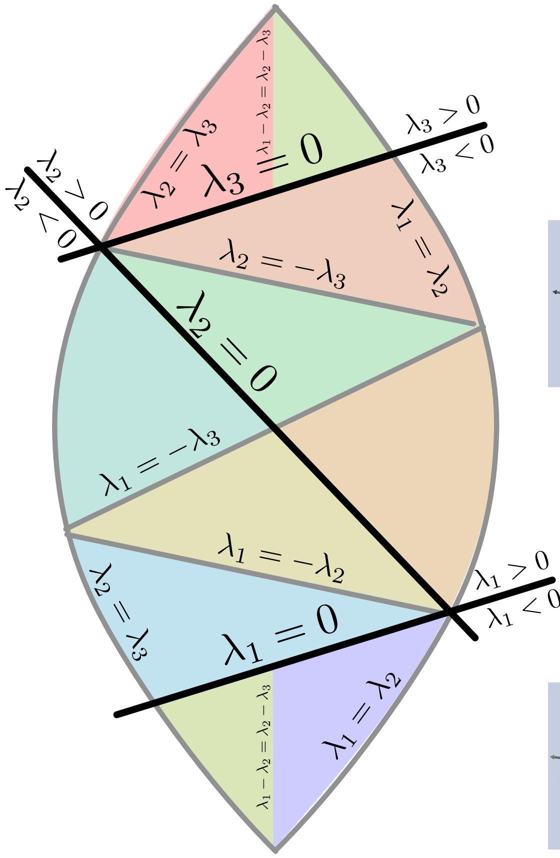




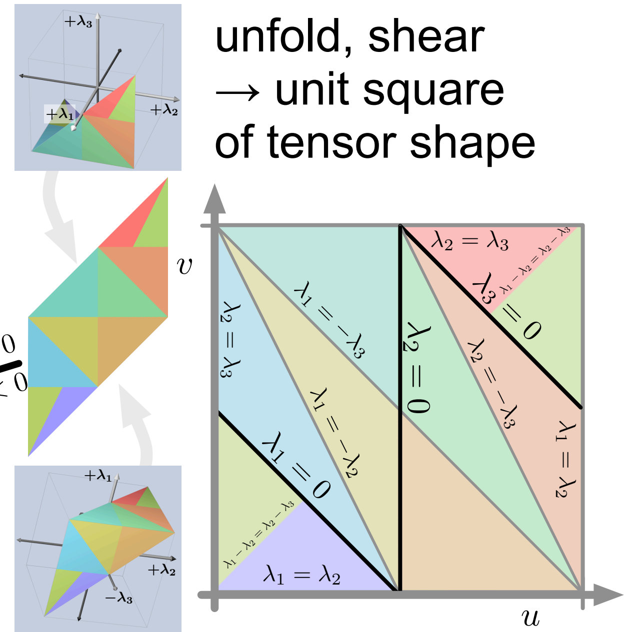
Kindlmann "Superquadric Tensor Glyphs", Vissym '04

Space of 3D tensor shape

Cubic projection

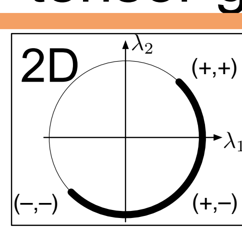


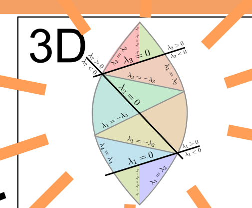
unfold, shear
→ unit square
of tensor shape



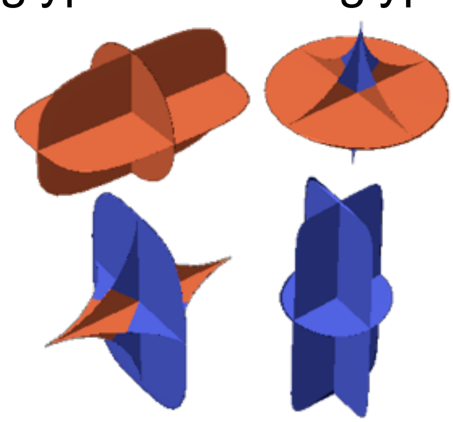
From 2D to 3D tensor glyphs

Understand space of 3D tensor shapes

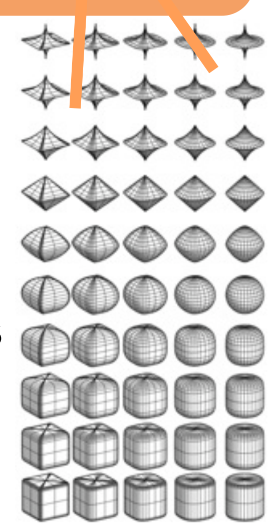




Scaffold (prototype) 3D glyphs with 2D glyphs

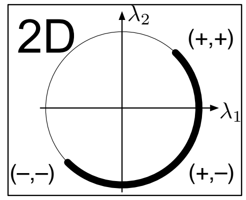


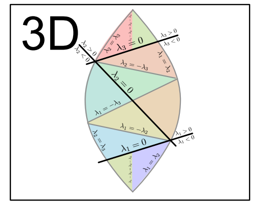
Find mapping from 3D tensor shapes to 3D superquadrics



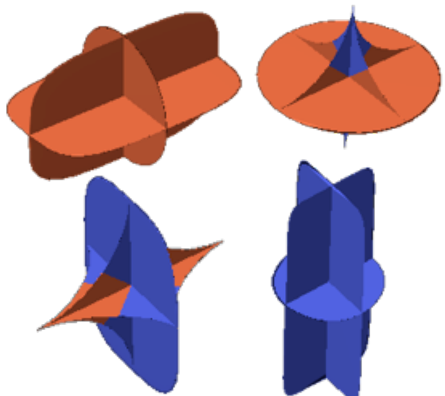
From 2D to 3D tensor glyphs

Understand space of 3D tensor shapes

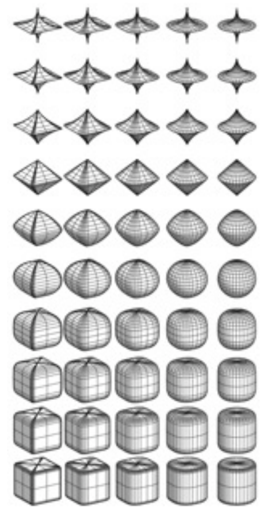


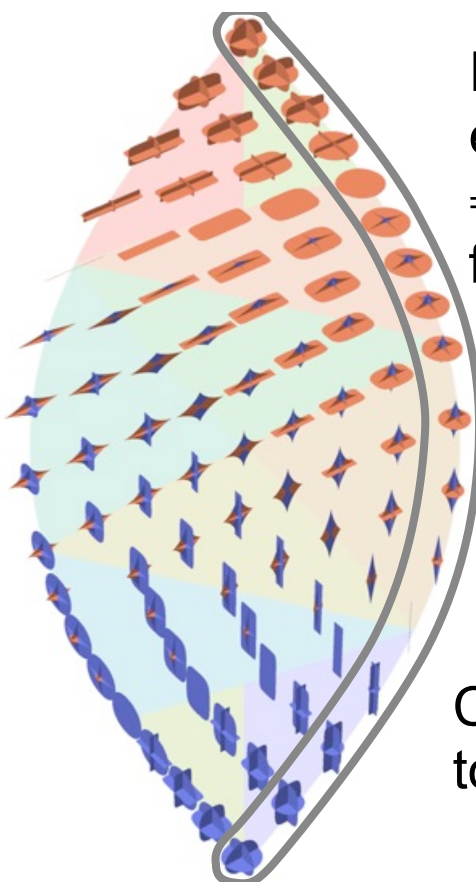


Scaffold (prototype) 3D glyphs with 2D glyphs

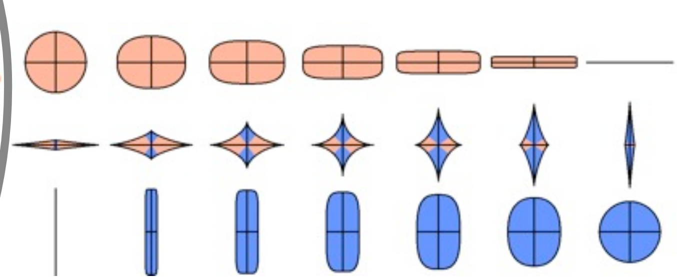


Find mapping from 3D tensor shapes to 3D superquadrics





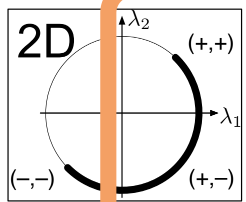
Invariance under eigenplane projection
⇒ orthogonal 2D glyphs form **scaffold** for 3D glyphs

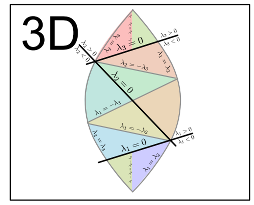


Can use these 3D shapes to guide next step ...

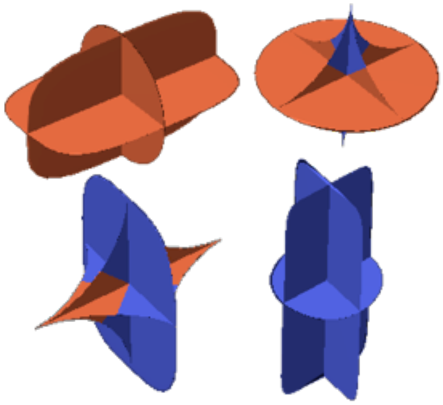
From 2D to 3D tensor glyphs

Understand space of 3D tensor shapes

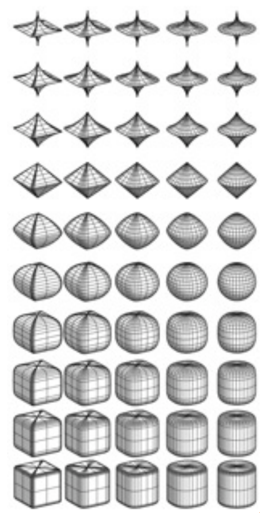


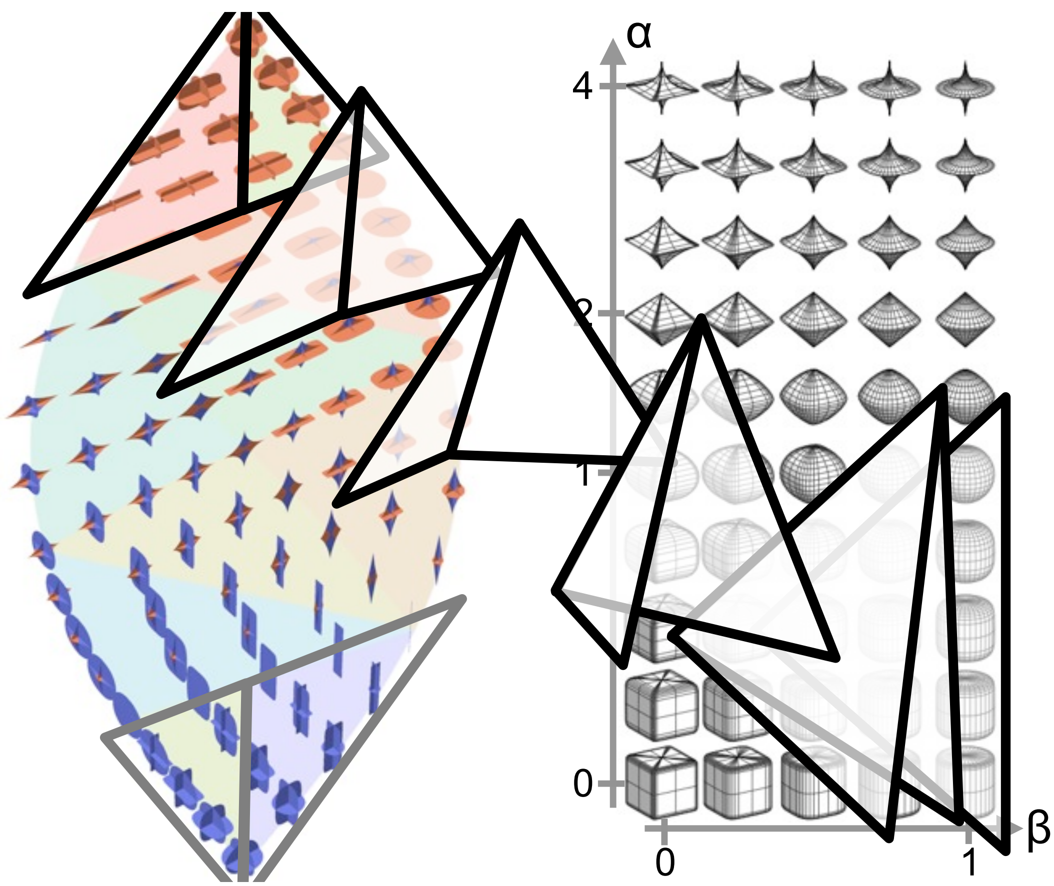


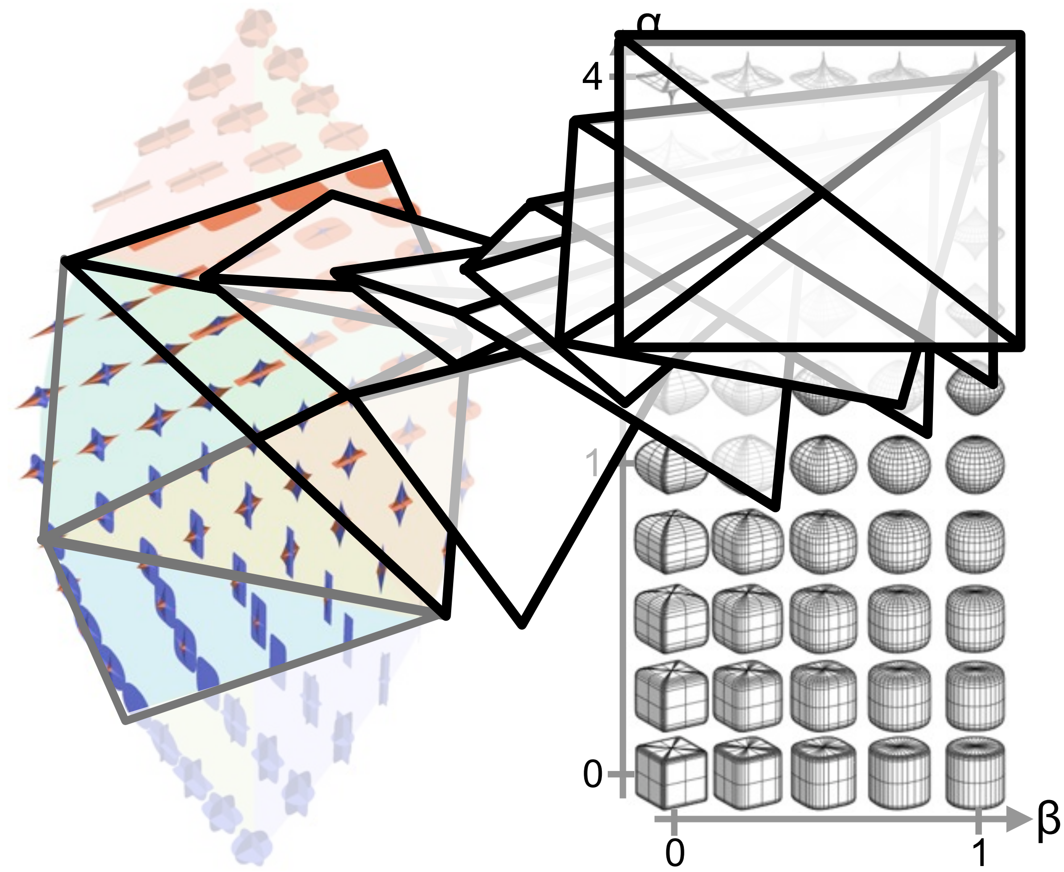
Scaffold (prototype) 3D glyphs with 2D glyphs

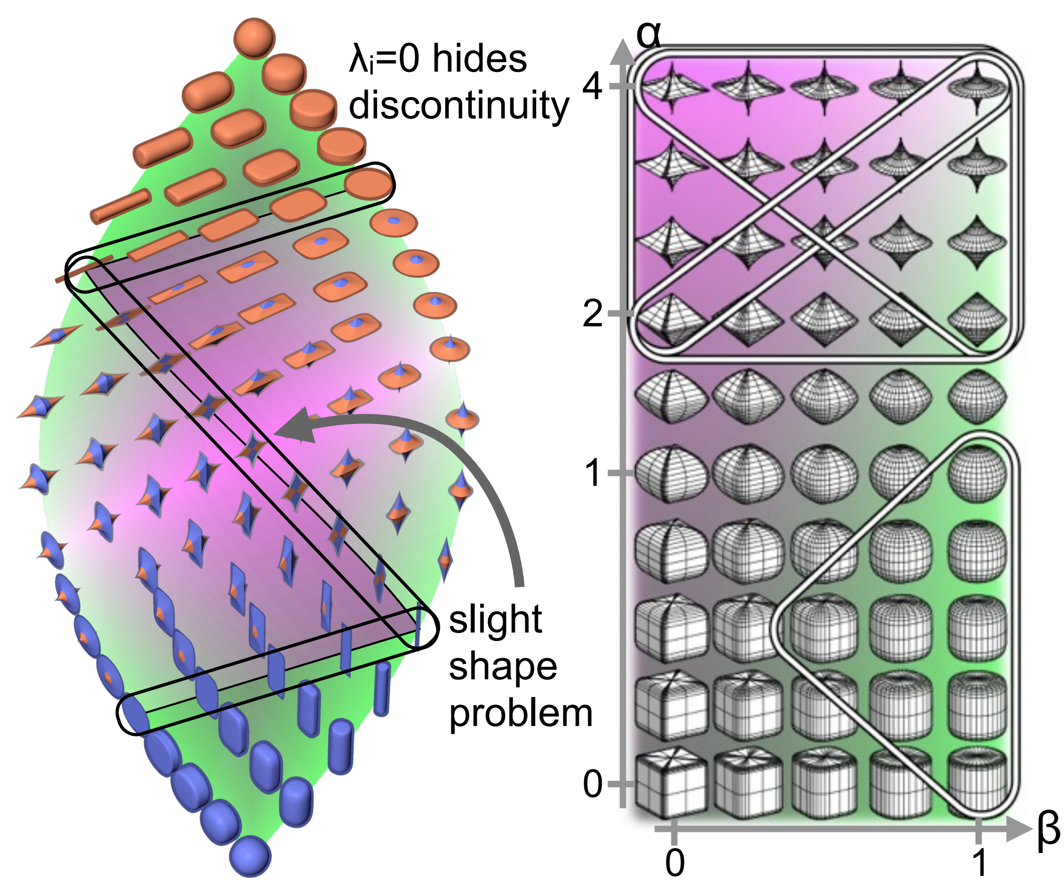


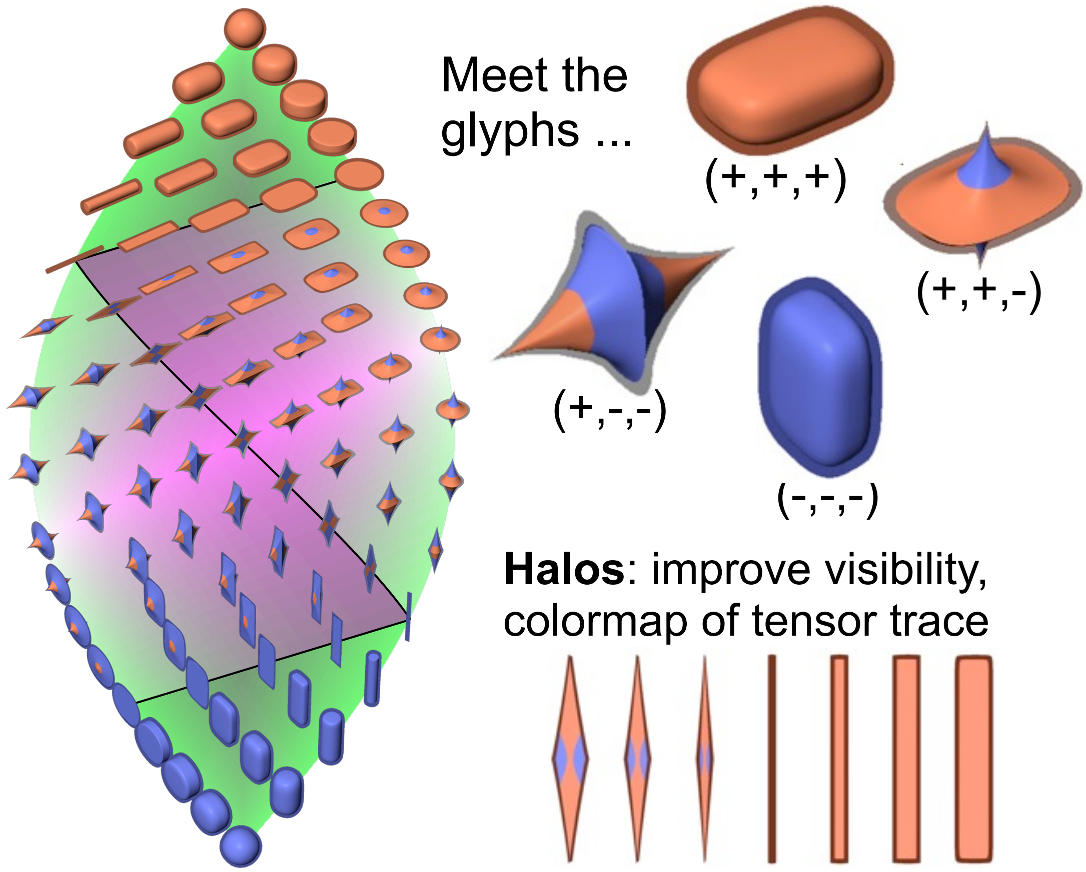
Find mapping from 3D tensor shapes to 3D superquadrics






α
4
1
0
β
0
1


λi=0 hides discontinuity
slight shape problem
α
4
2
1
0
β
0
1



The Glyph Rendering Pipeline

$$G(\mathbf{D}) = \underbrace{s(\|\mathbf{D}\|)\,\mathbf{R}\tilde{\Lambda}}\,\underbrace{B(\tilde{\lambda}_i)}$$

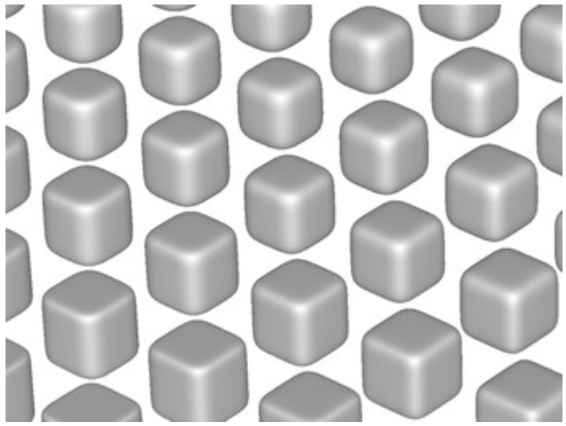
4. Scale and rotate (modelview matrix)

1. Map λ_i to (u,v)
2. Map (u,v) to (α,β)
3. Generate Base Glyph

5. Color code sign (fragment shader)

Using a Palette of Base Geometry

Precompute and re-use superquadric geometry on a grid in (α,β) space



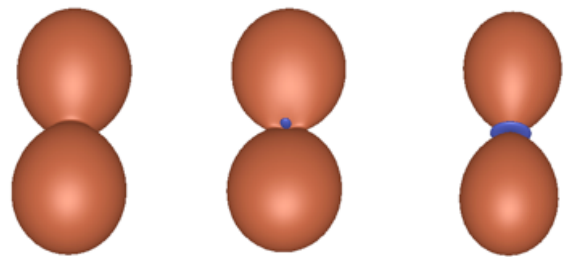
Implementation & Performance

- Geometry-based implementation
- Performance:
  - 3000 full-resolution glyphs
  - 1800x1000 viewport
  - 25fps (incl. halos) on NVIDIA Quadro FX 1800

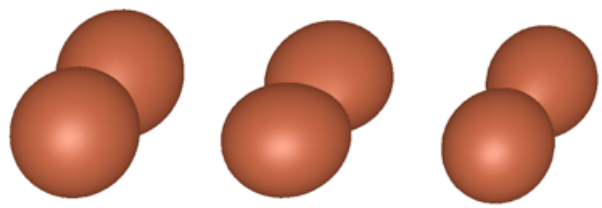
Most code is in Teem; please see tutorial on http://www.ci.uchicago.edu/~schultz/sphinx/

Why not use Reynolds Glyphs?

Colored Reynolds Glyphs clearly show negative eigenvalues…

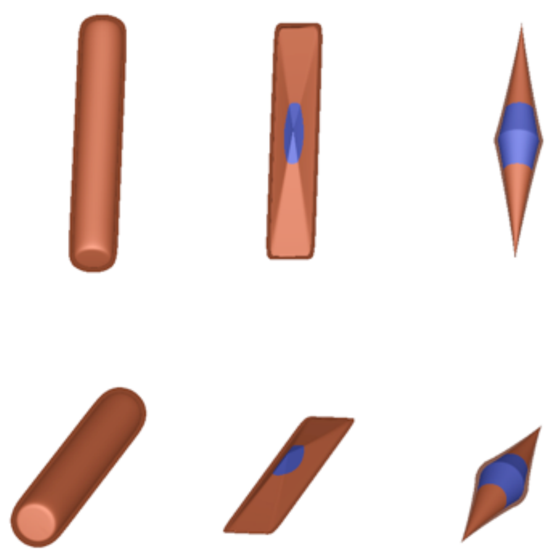


…unless you're viewing them from an unfortunate direction.

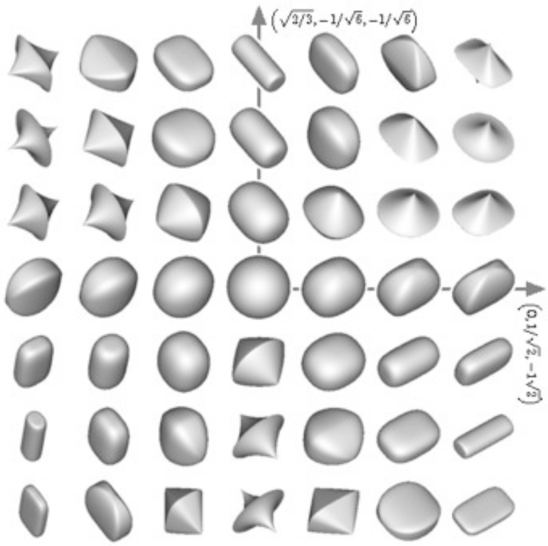


Why Prefer Superquadrics?

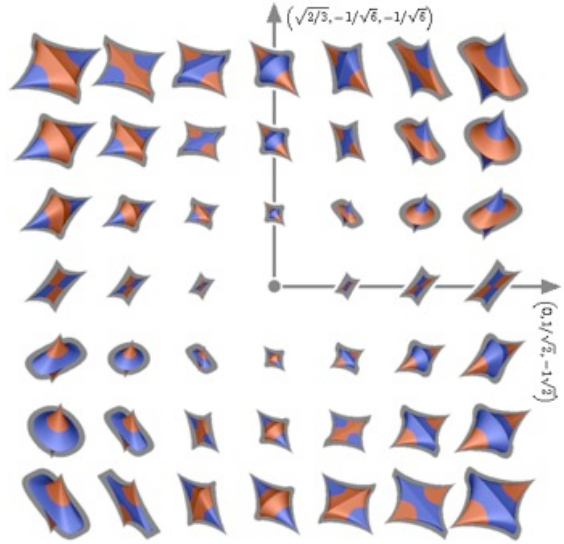
Superquadrics remain legible from a wide range of directions.



What About Traceless Tensors?

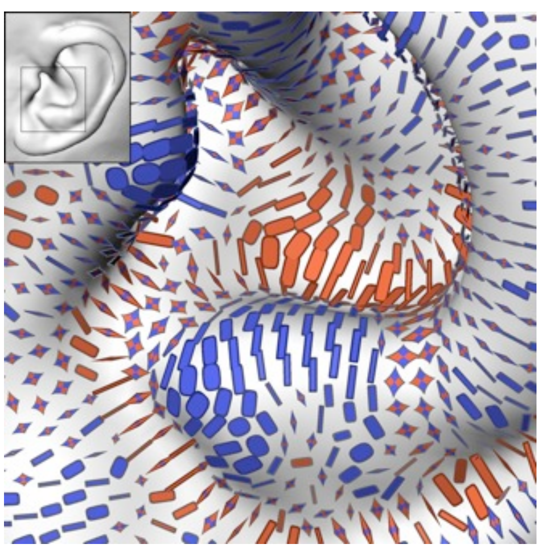


Existing Traceless
Superquadric Glyphs
[Jankun-Kelly/Mehta, Vis06]

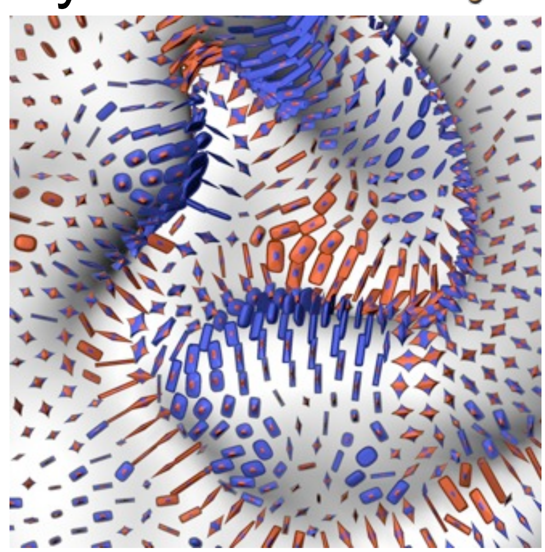


Our new
Superquadric Glyphs

Hessians vs. Geometry Tensors



Geometry Tensors

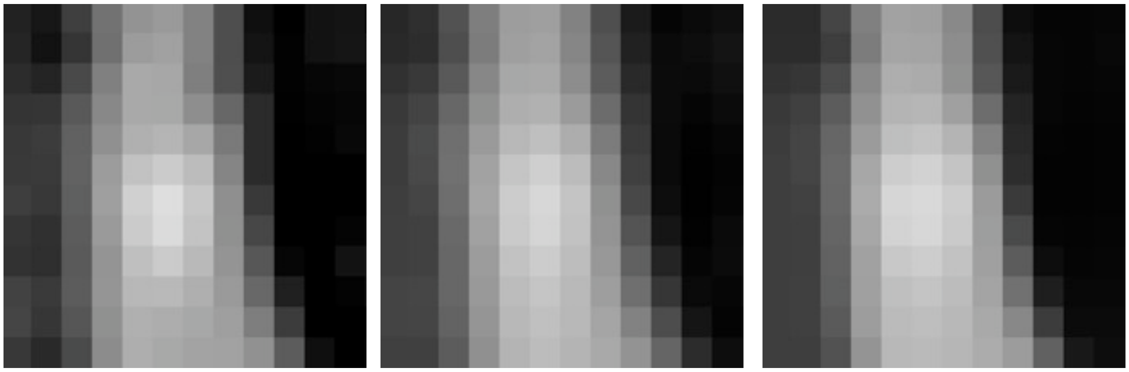


Hessians

See how geometry tensors are
related to Hessians

$\mathbf{G} = (\mathbf{I}-\mathbf{n}\mathbf{n}^T)\mathbf{H}(\mathbf{I}-\mathbf{n}\mathbf{n}^T)/||\mathbf{g}||$

Inspecting Smoothing Image Filters

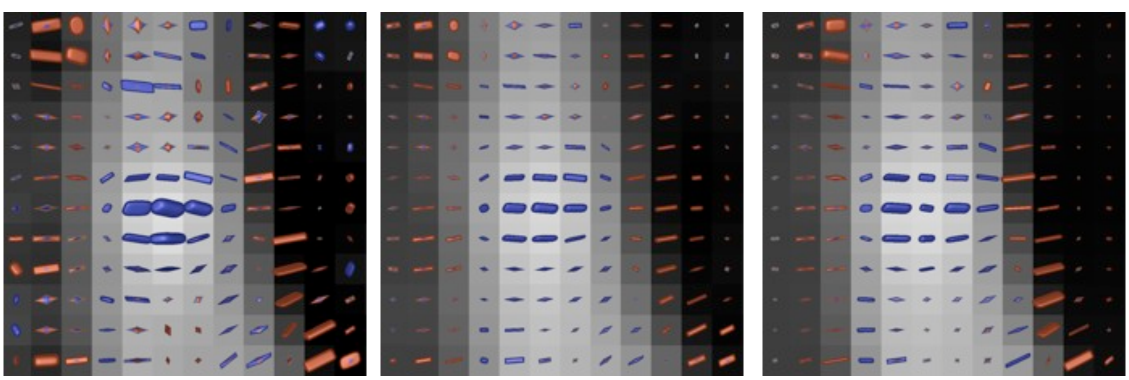


Original Data | Gaussian Smoothing | Total Variation Flow

Equal Remaining Variance

What are Differences?

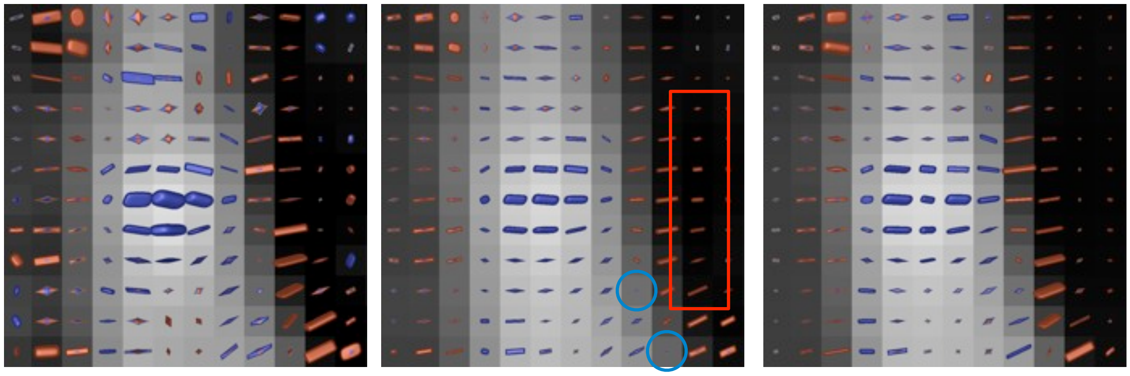
Inspecting Smoothing Image Filters



Original Data | Gaussian Smoothing | Total Variation Flow

Let’s look at Hessians…

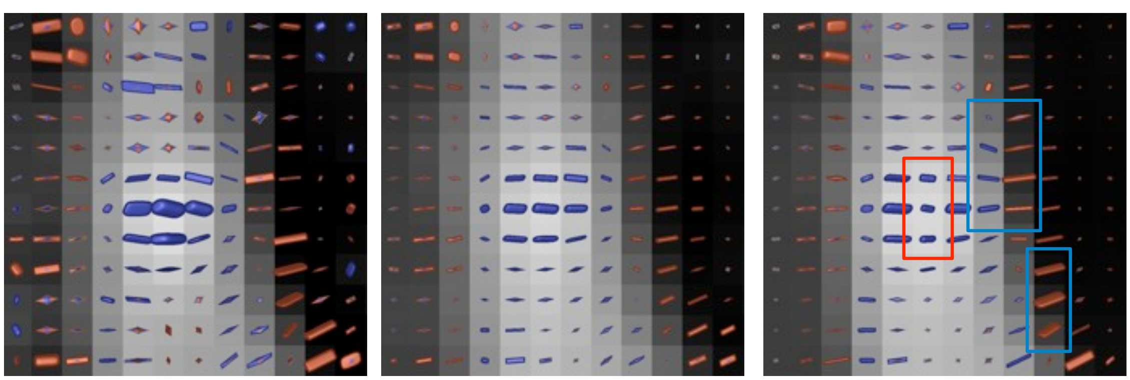
Inspecting Smoothing Image Filters



Original Data Gaussian Smoothing Total Variation Flow

Gaussian Filtering **Smoothes Hessians**.
Leads to “smearing out” into flat regions
and cancellation effects.

Inspecting Smoothing Image Filters



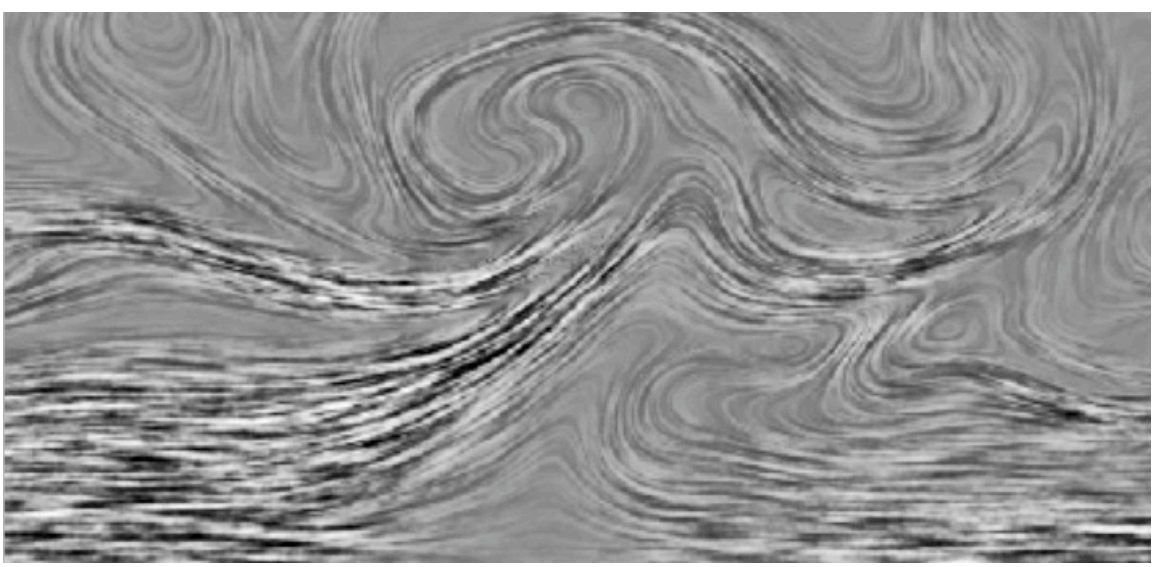
Original Data Gaussian Smoothing Total Variation Flow

TV flow creates flat regions
and sharp edges between them.

Results, turbulent jet flow

LIC

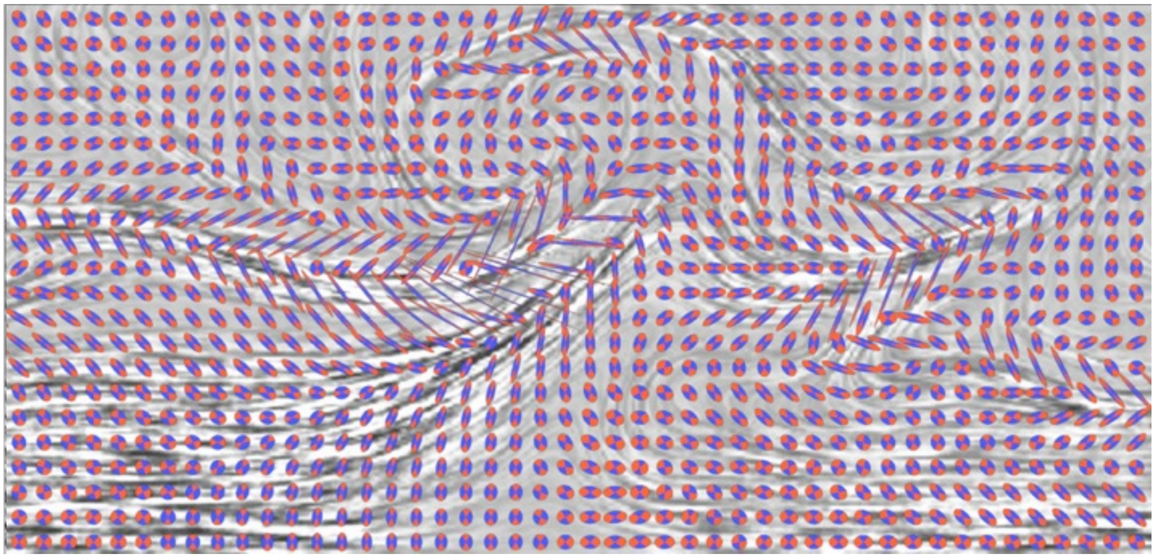
Results, turbulent jet flow



LIC, modulate contrast by velocity

Results, turbulent jet flow

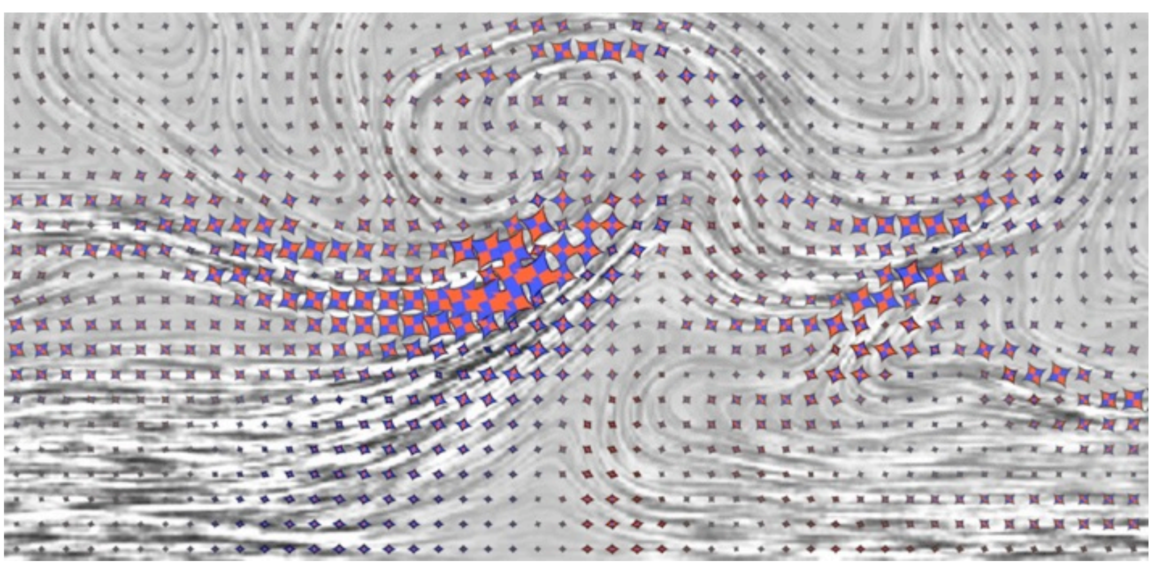
Rate-of-Deformation Tensor = $(\nabla\mathbf{v} + \nabla^T\mathbf{v})/2$



Ellipses with λ → exp(λ)

Results, turbulent jet flow

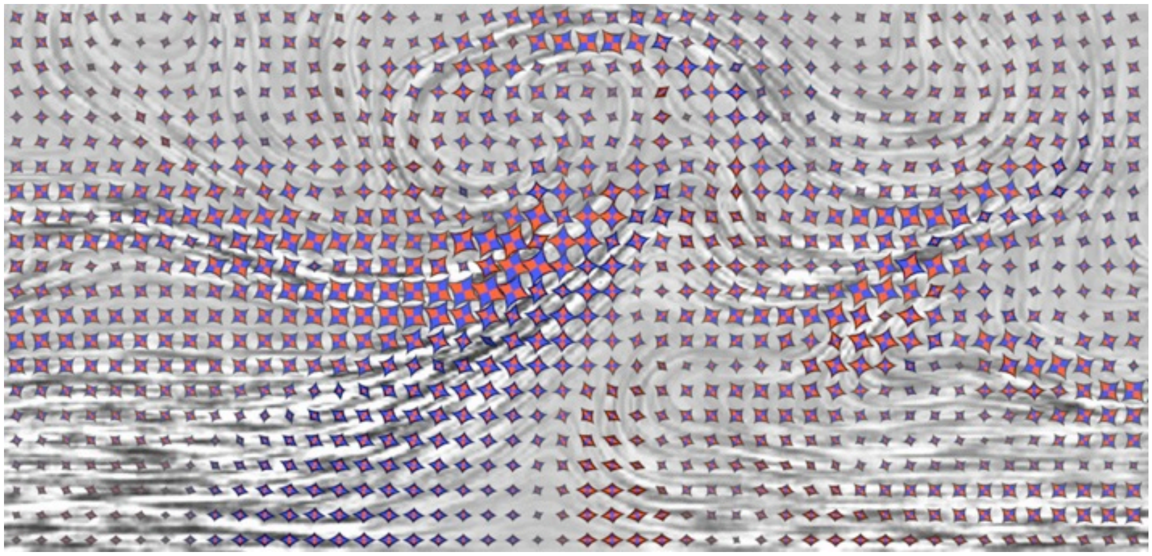
Rate-of-Deformation Tensor = $(\nabla\mathbf{v} + \nabla^T\mathbf{v})/2$



New glyphs, $s(\|\mathbf{D}\|) \propto \|\mathbf{D}\|$

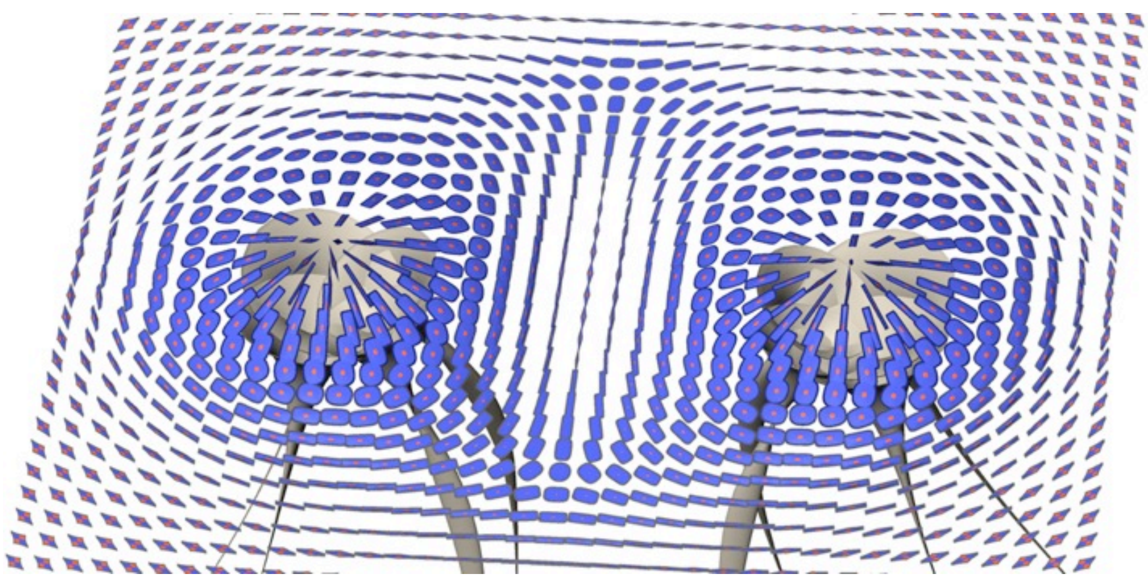
Results, turbulent jet flow

Rate-of-Deformation Tensor = $(\nabla \mathbf{v} + \nabla^T \mathbf{v})/2$



New glyphs, $s(\|\mathbf{D}\|) \propto \|\mathbf{D}\|^{1/2}$

Results, double point stress field



Can now see tensors underlying hyperstreamlines $s(\|\mathbf{D}\|) \propto \|\mathbf{D}\|^{1/10}$

Conclusions

- Presented a new set of tensor glyphs
  - Can visualize all symmetric 3D 2nd-order tensors
- Glyphs design guided by principles
  - Need disambiguity, avoid misleading symmetries
  - Eigenplane projection invariance ⇒ **scaffolds**
- Future: still no satisfactory glyphs for:
  - General 2nd-order (non-symmetric) tensors
  - Rotations

Acknowledgements

- Funding: DAAD
- Flow data: Wolfgang Kollmann, UC Davis
- Symmetry and Continuity discussion:

2009 Dagstuhl Scientific Visualization Seminar 09251

Thank you