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# Glyphs for Asymmetric Second-Order 2D Tensors

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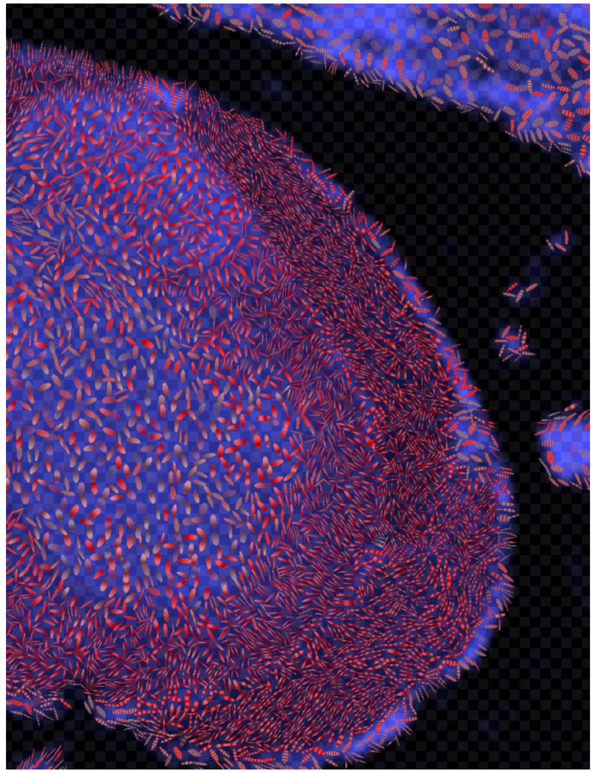
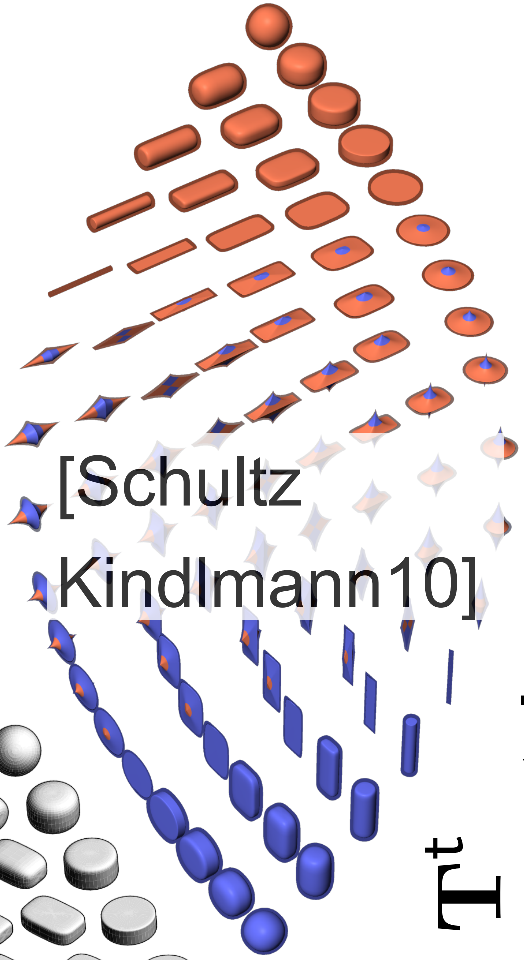
# Why another tensor glyph?

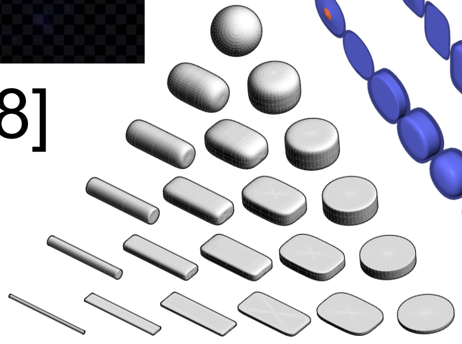
- Tensors locally model complex physical phenomena
- Intrinsically multi-variate: focus of vis research



[Laidlaw+98]



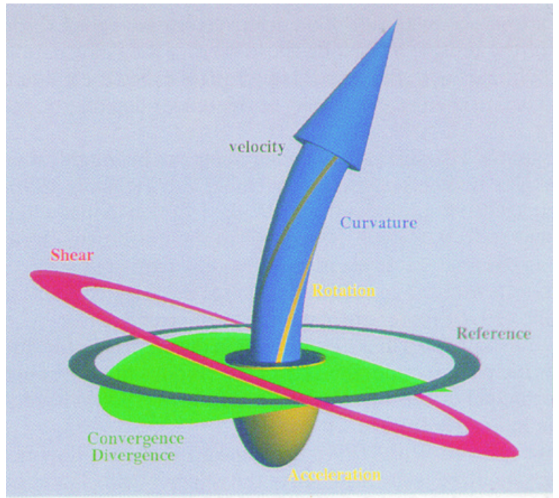
[Schultz Kindlmann10]



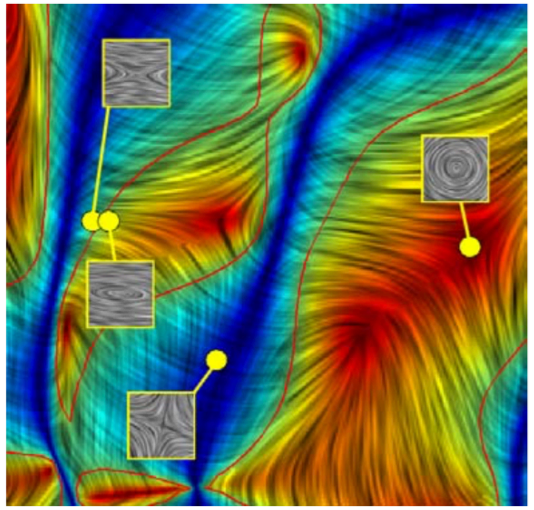
[Kindlmann04]

$T = T^t$ symmetric

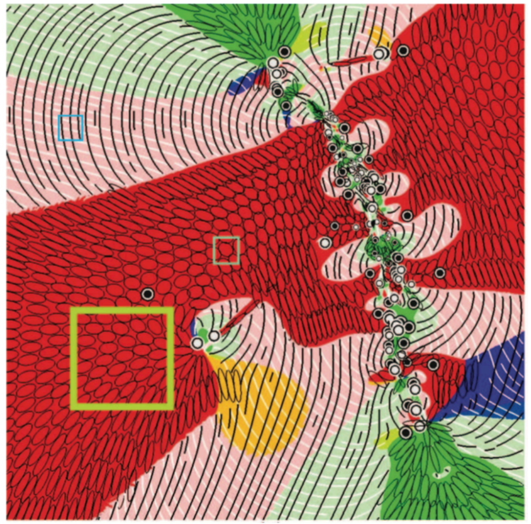
asymmetric $T \neq T^t$



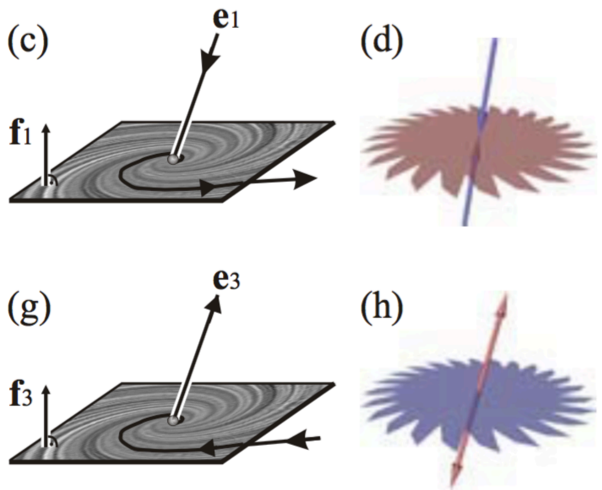

[deLeeuw vanWijk93]



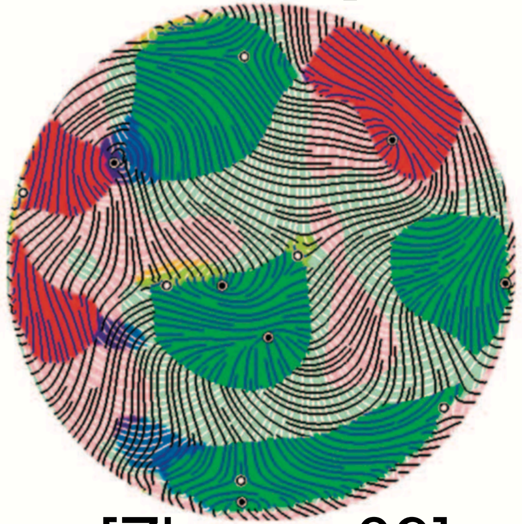
[ZhengPang05]



[Chen+11]




[Theisel+03]



[Zhang+09]

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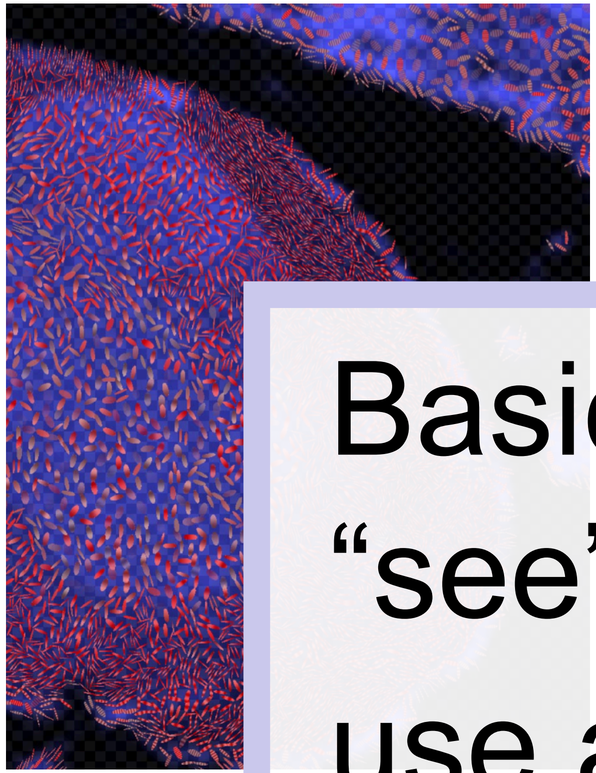

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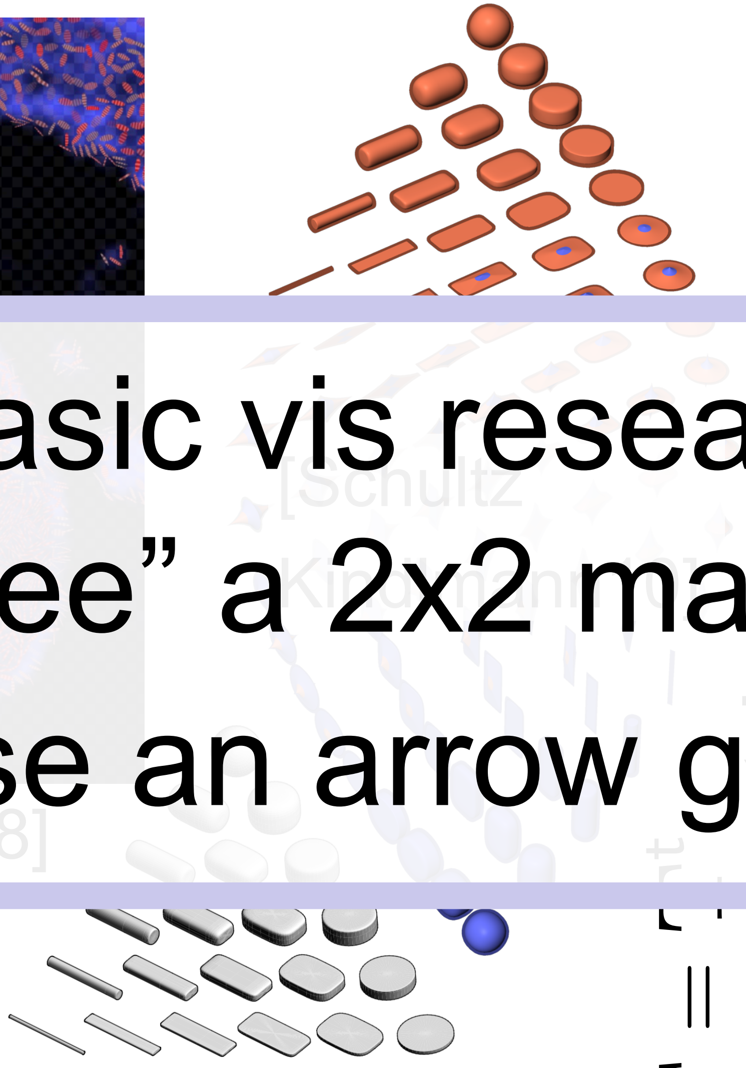

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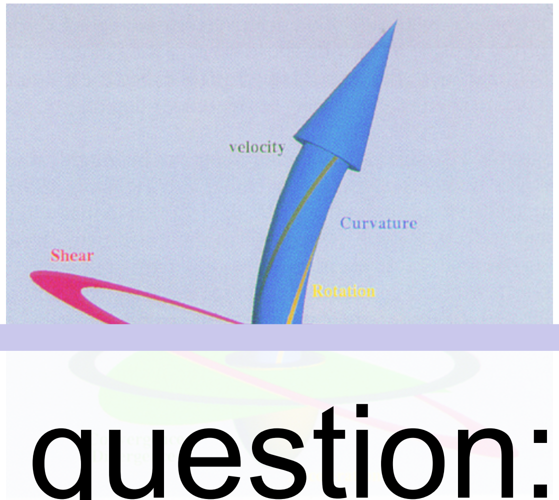


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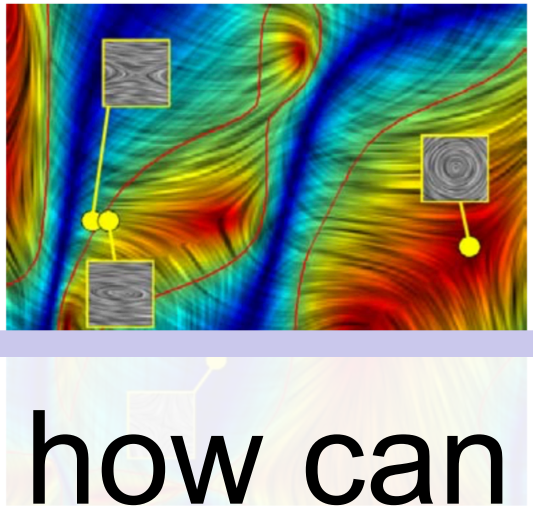

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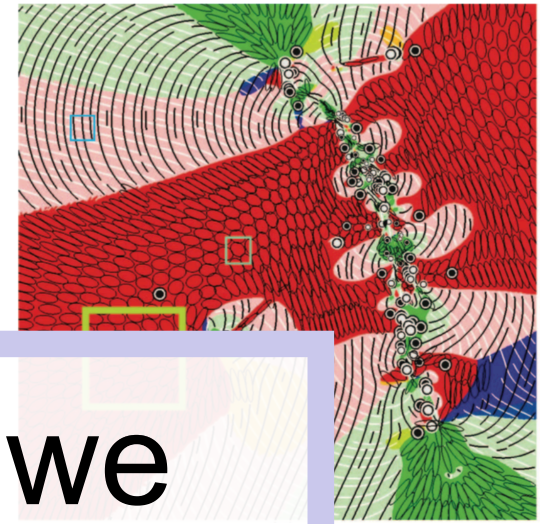

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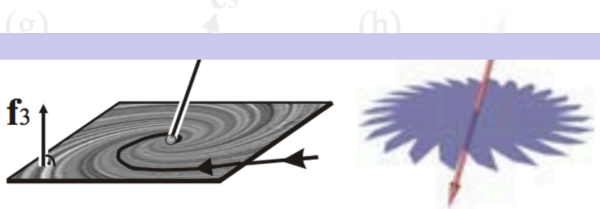


[Theisel+03]

[Zhang+09]

Basic vis research question: how can we “see” a 2x2 matrix as clearly as we can use an arrow glyph ↗ to “see” a 2-vector?

# Three-way decomposition of tensor

$$\mathbf{T}$$

symmetric

$$\frac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathrm{t}})$$

isotropic

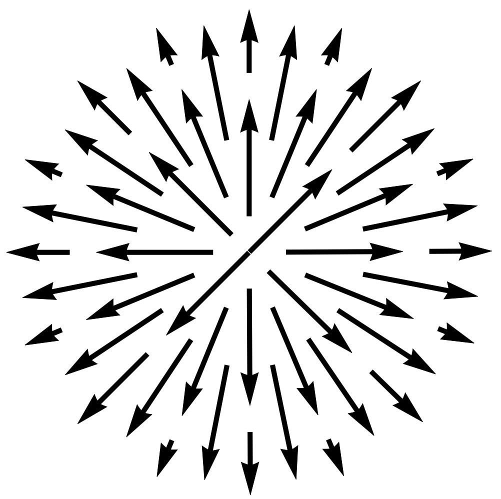
$$\frac{1}{2}\,\mathrm{tr}(\mathbf{T})\mathbf{I}$$

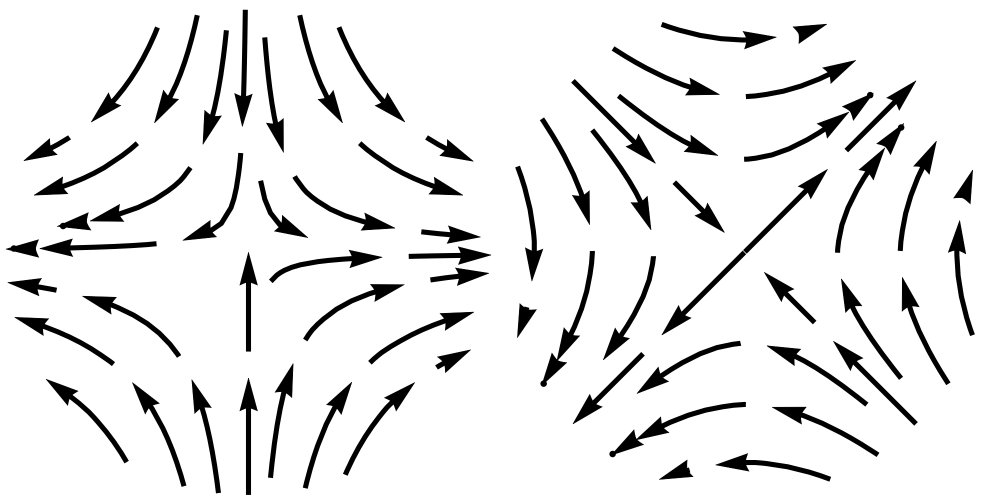
anisotropic

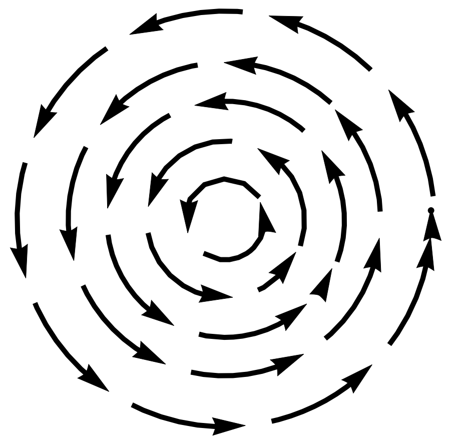
$$\frac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathrm{t}}) - \frac{1}{2}\,\mathrm{tr}(\mathbf{T})\mathbf{I}$$

anti-symmetric

$$\frac{1}{2}(\mathbf{T} - \mathbf{T}^{\mathrm{t}})$$

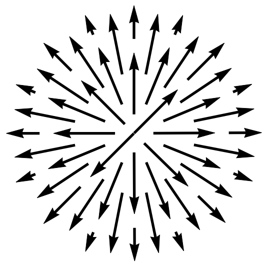
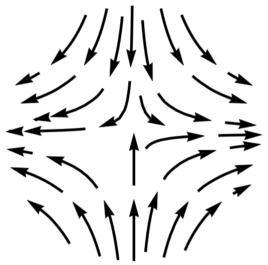
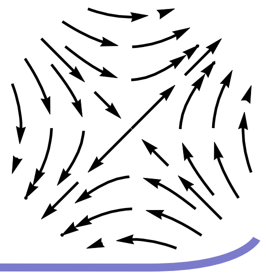
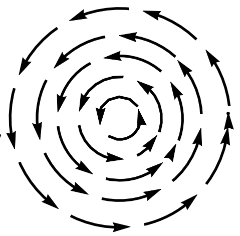






reference visualization of $\mathbf{T}$: streamlines through $\mathbf{v}(\mathbf{x}) = \mathbf{T}\mathbf{x}$

# Decomposition,coords of tensor in new basis

$\mathbf{B}_D$  $\mathbf{B}_{S_1}$  $\mathbf{B}_{S_2}$  $\mathbf{B}_R$ 

**D**ilation/Contraction | **S**tretching or **S**train rate | **R**otation

components: $D, S_1, S_2, R;\; X(\mathbf{T}) = \mathbf{T} : \mathbf{B}_X = \mathrm{tr}(\mathbf{T}^{\mathrm{t}}\mathbf{B}_X)$

$\mathbf{B}_{S_1}, \mathbf{B}_{S_2}$ rotations of each other:

$$\mathbf{A}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$S = \sqrt{S_1^2 + S_2^2} \qquad \mathbf{B}_{S_2} = \mathbf{A}(\pi/4)\mathbf{B}_{S_1}\mathbf{A}(-\pi/4)$$

Coordinates: $(D, S, R, \alpha) : \mathbf{T} = D\,\mathbf{B}_D + S\,\mathbf{A}(\alpha)\mathbf{B}_{S_1}\mathbf{A}(\alpha)^{\mathrm{t}} + R\,\mathbf{B}_R$

# §3: Relative to previous math

- $(D,S,R) = \sqrt{2}(\gamma_d,\ \gamma_s,\ \gamma_r)$ [Zhang+09] [Chen+11]
- Highlighting connection to invariants and norms:

$$\mathrm{tr}(\mathbf{T})=\sqrt{2}D;\ \det(\mathbf{T})=(D^2-S^2+R^2)/2;\ ||\mathbf{T}||_F=\sqrt{(D^2+S^2+R^2)}$$

- Using double-angle formulae for stretch component

$$\begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix}=\sqrt{2}\mathbf{A}(\theta/2)\mathbf{B}_{S_1}\mathbf{A}(-\theta/2);\ \ \mathbf{A}(\alpha)=\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}$$

- Simplifies expression of dual [ZhengPang05], pseudo-eigenvectors [Zhang+09] [Chen+11]

# Glyph formation and design principles

$$G(\mathbf{T}) = s(\|\mathbf{T}\|_F)\ \tilde{\mathbf{E}}\ \tilde{\Lambda}\ \mathbf{b}(D, S, R)$$

[SchultzKindlmann10]

over-all scaling (s); transform ($\tilde{\mathbf{E}}$); diagonal matrix to scale along each axis ($\tilde{\Lambda}$); base geometry ($\mathbf{b}$)

- Applying Algebraic Visualization Design [KindlmannScheidegger14], especially **Principle of Visual-Data Correspondence**:
- Continuity: $T_1 \approx T_2 \Rightarrow G(T_1) \approx G(T_2)$
- Symmetry preservation: $T = RTR^{-1} \Rightarrow G(T) = R(G(T))$
  - S=0 means b should ideally be continuously rotationally symmetric
- "Transform legibility": rotation: $G(RTR^{-1}) = R(G(T))$; scaling: $G(sT) =$ "$s(G(T))$"; negation $G(-T) =$ "$-G(T)$"

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over-all scaling; transform; diagonal matrix to scale along each axis; base geometry; 4D to 3D space

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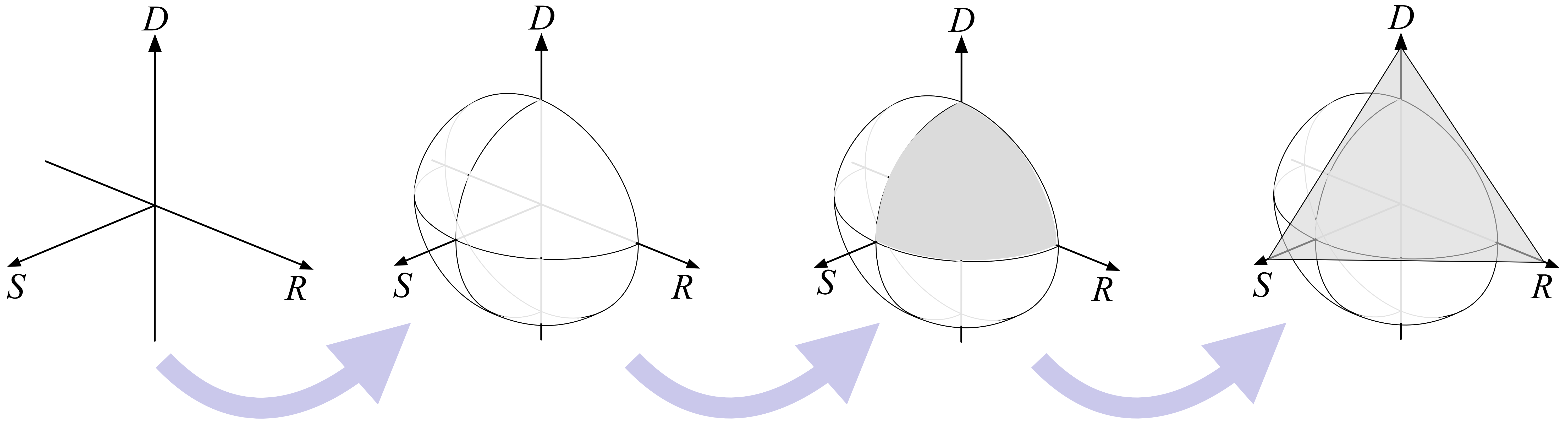
# Simplifying the glyph design space




Scale legibility:
design $\mathbf{b}$ for
$||\mathbf{T}||_F = 1$

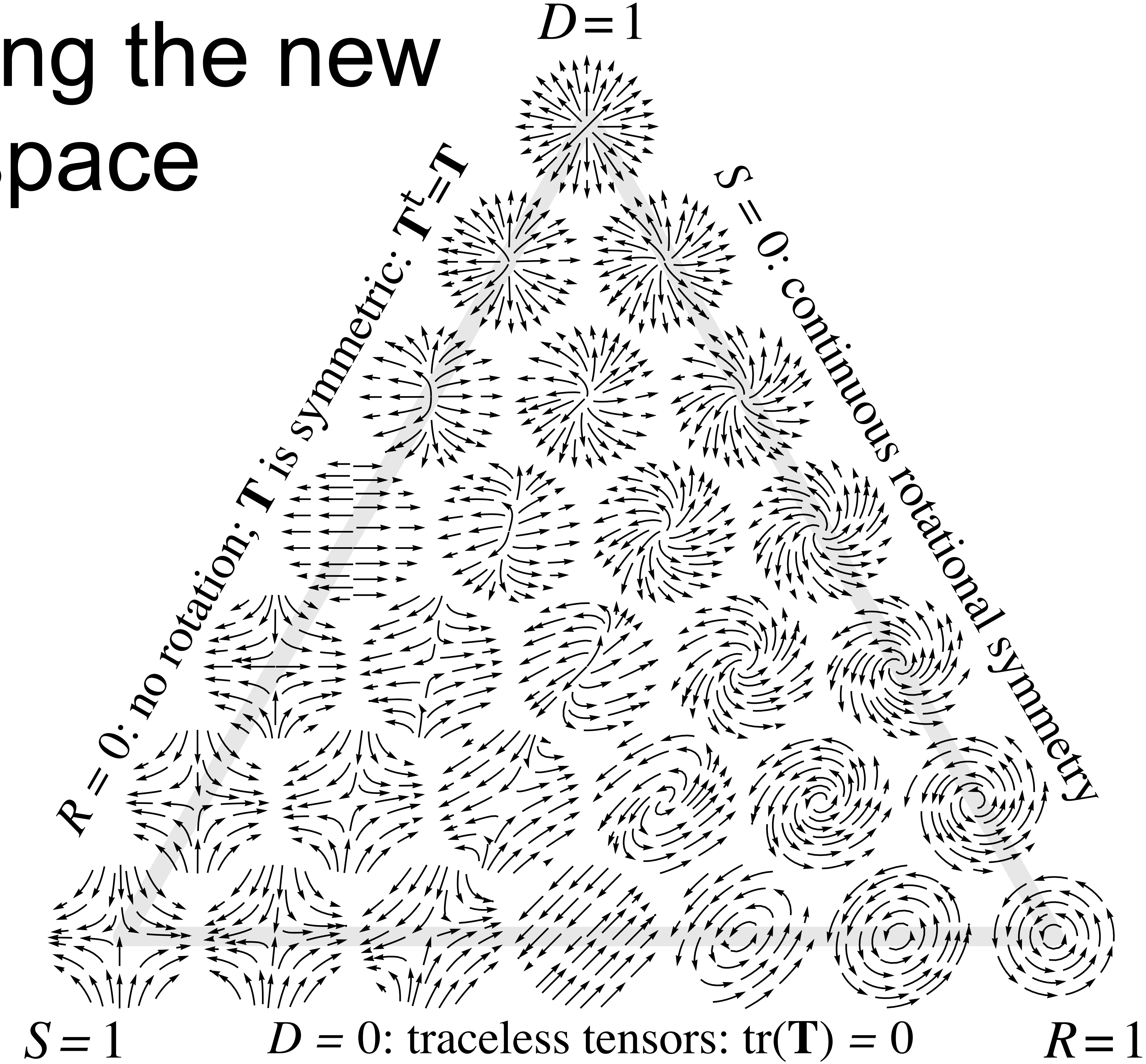
Negation legibility:

$$G(-D,S,R) = -G(D,S,R)$$

$$G(D,S,-R) = -G(D,S,R)$$

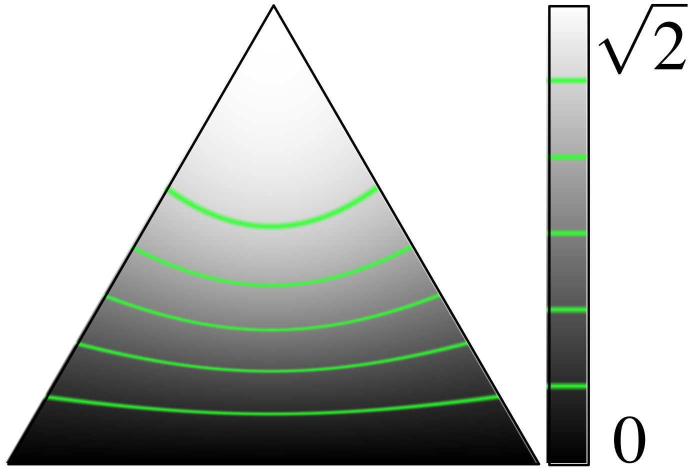
Gnomonic
projection

# Visualizing the new design space



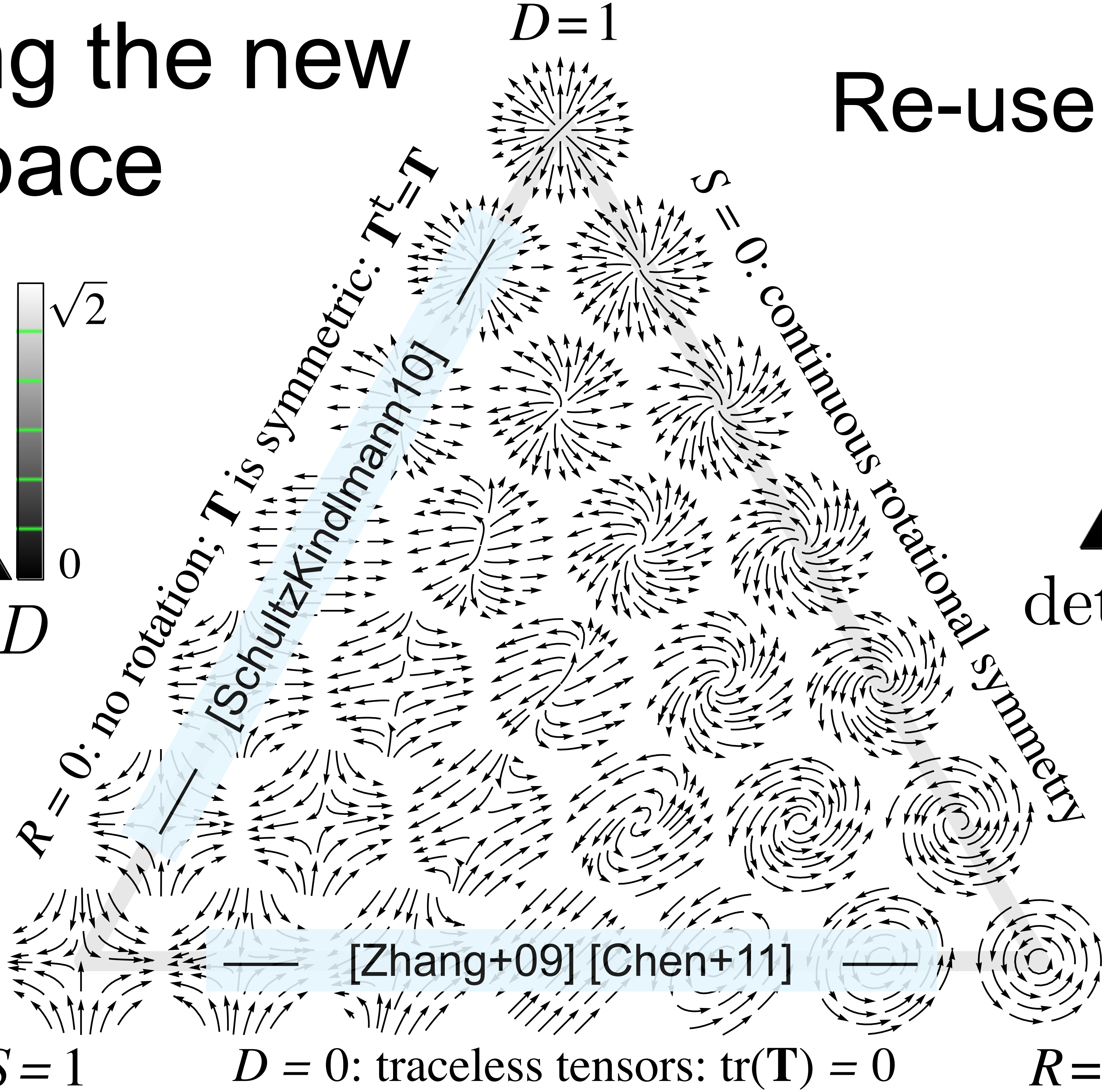

Shown with streamlines through $\mathbf{v}(\mathbf{x}) = \mathbf{T}\mathbf{x}$

# Visualizing the new design space

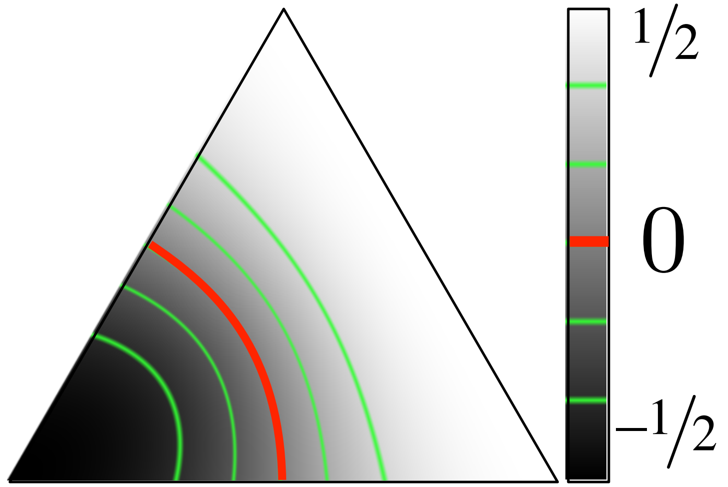



$\mathrm{tr}(\mathbf{T}) = \sqrt{2}D$

Shown with streamlines through $\mathbf{v}(\mathbf{x}) = \mathbf{T}\mathbf{x}$



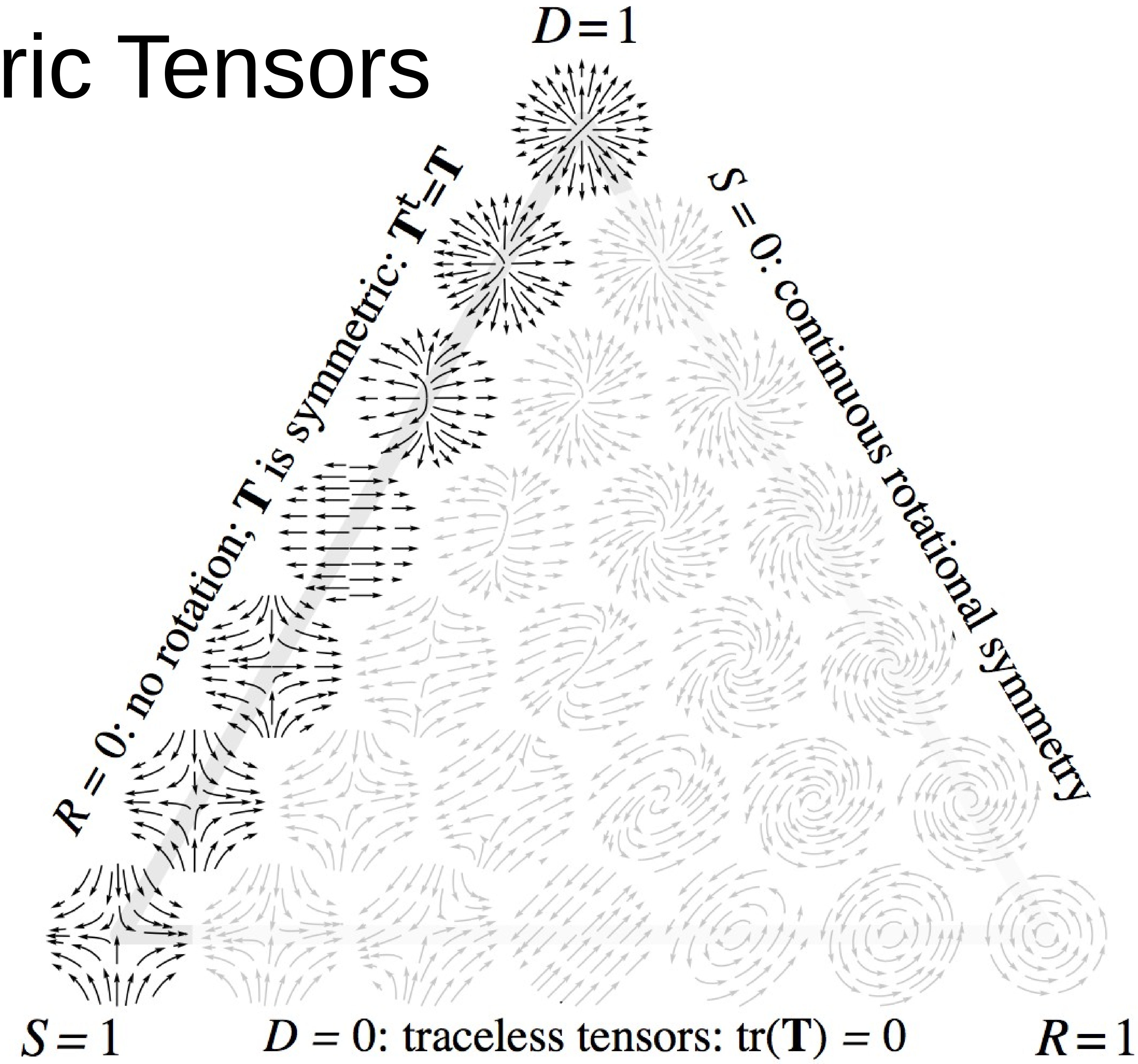

# Re-use this space !



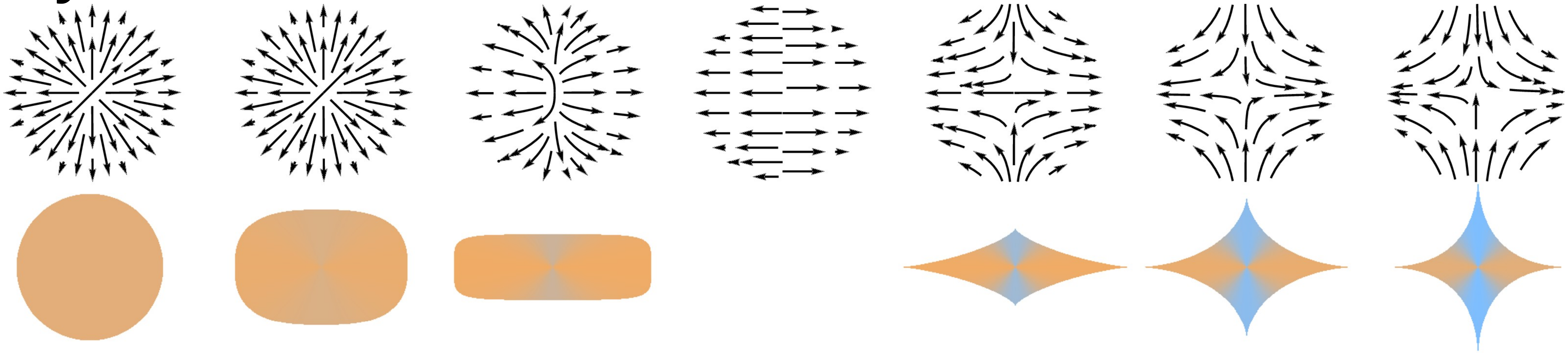

$\det(\mathbf{T}) = \frac{1}{2} - S^2$

# Our strategy: design on edges, then fill interior

# Symmetric Tensors

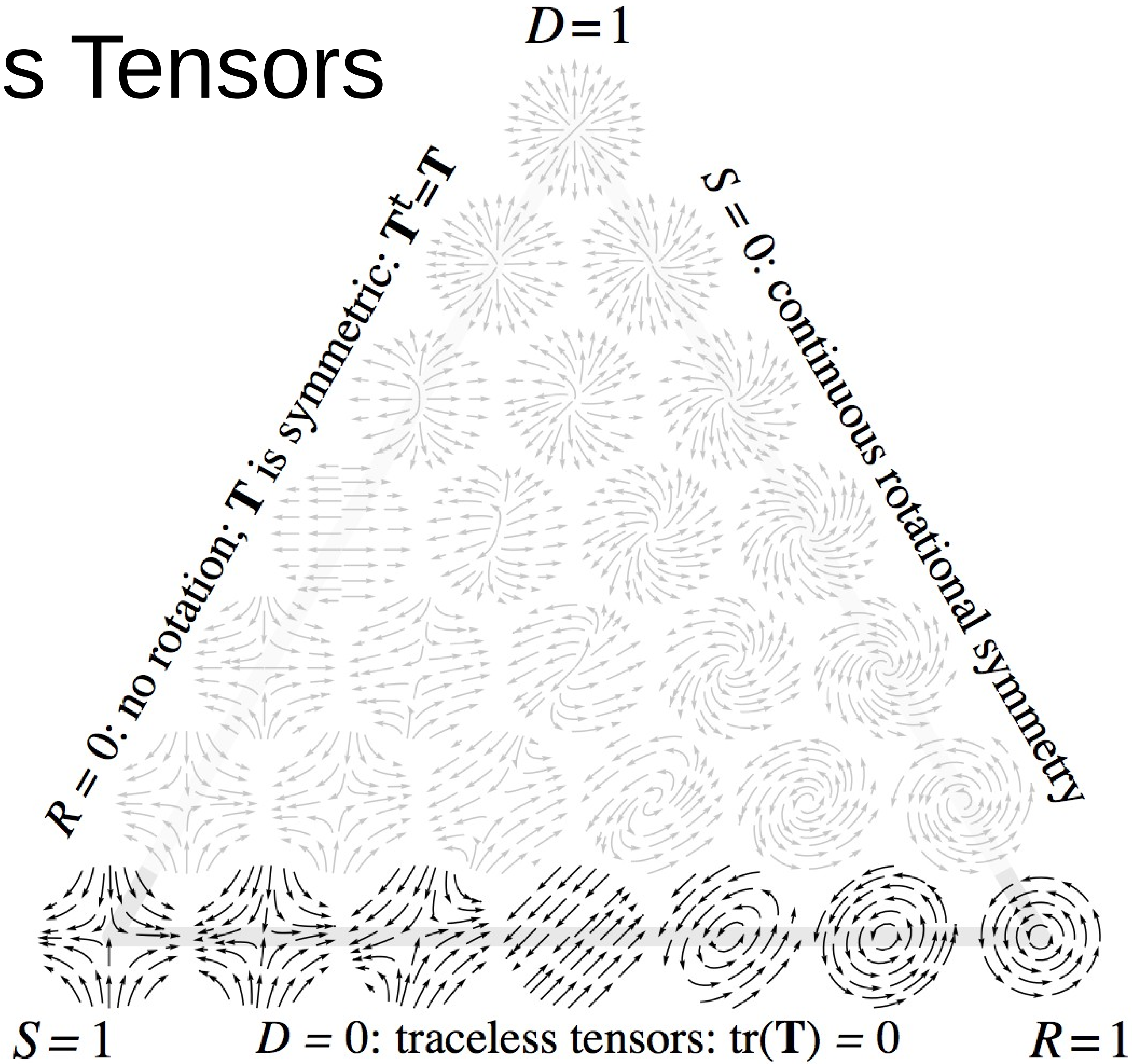


# Symmetric Tensors



- Superquadrics: $(\cos^a \theta, \sin^a \theta)$
- Exponent varied to indicate rotational asymmetry
- Axes scaled by eigenvalue
- Negation legibility: negate A,B in CIELAB
  - (L, A, B) = (80, 5.8 x·T·x, 23.2 x·T·x)

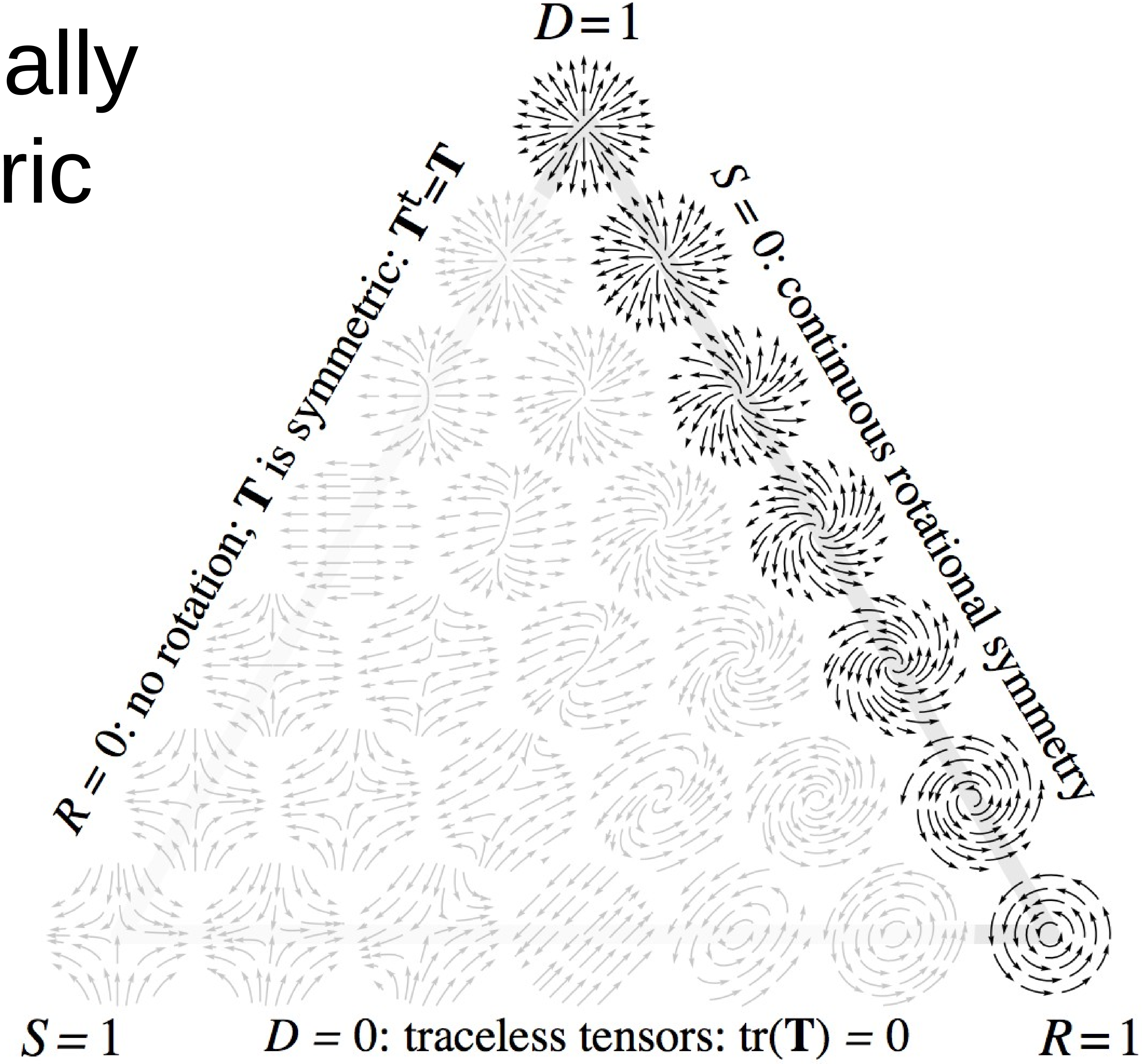
# Traceless Tensors



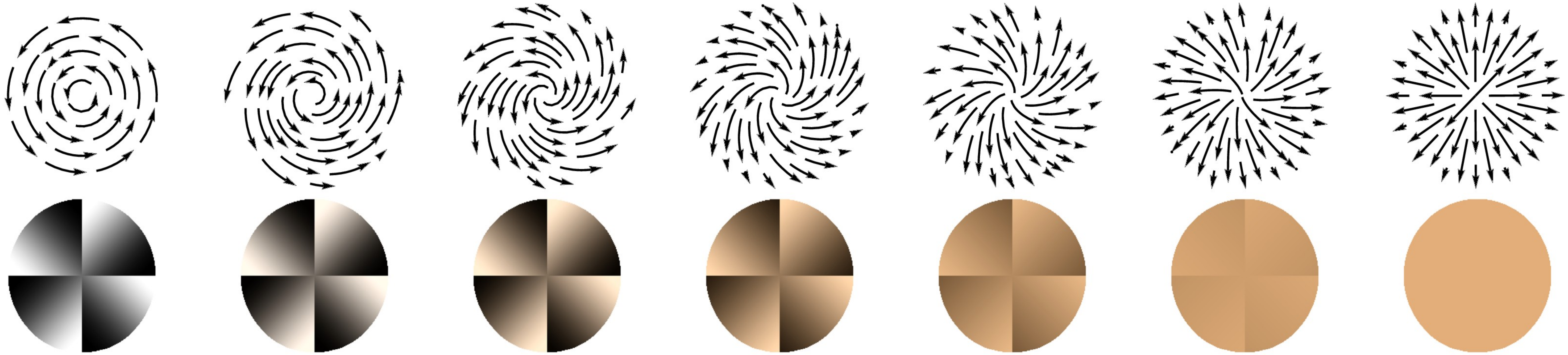
# Traceless Tensors

- Deform with matrix of eigenvector columns when real
- Use matrix of pseudoeigenvectors [Zhang+09] when complex
- Same coloring as in symmetric case: generates solid gray for S=0

# Rotationally Symmetric Tensors



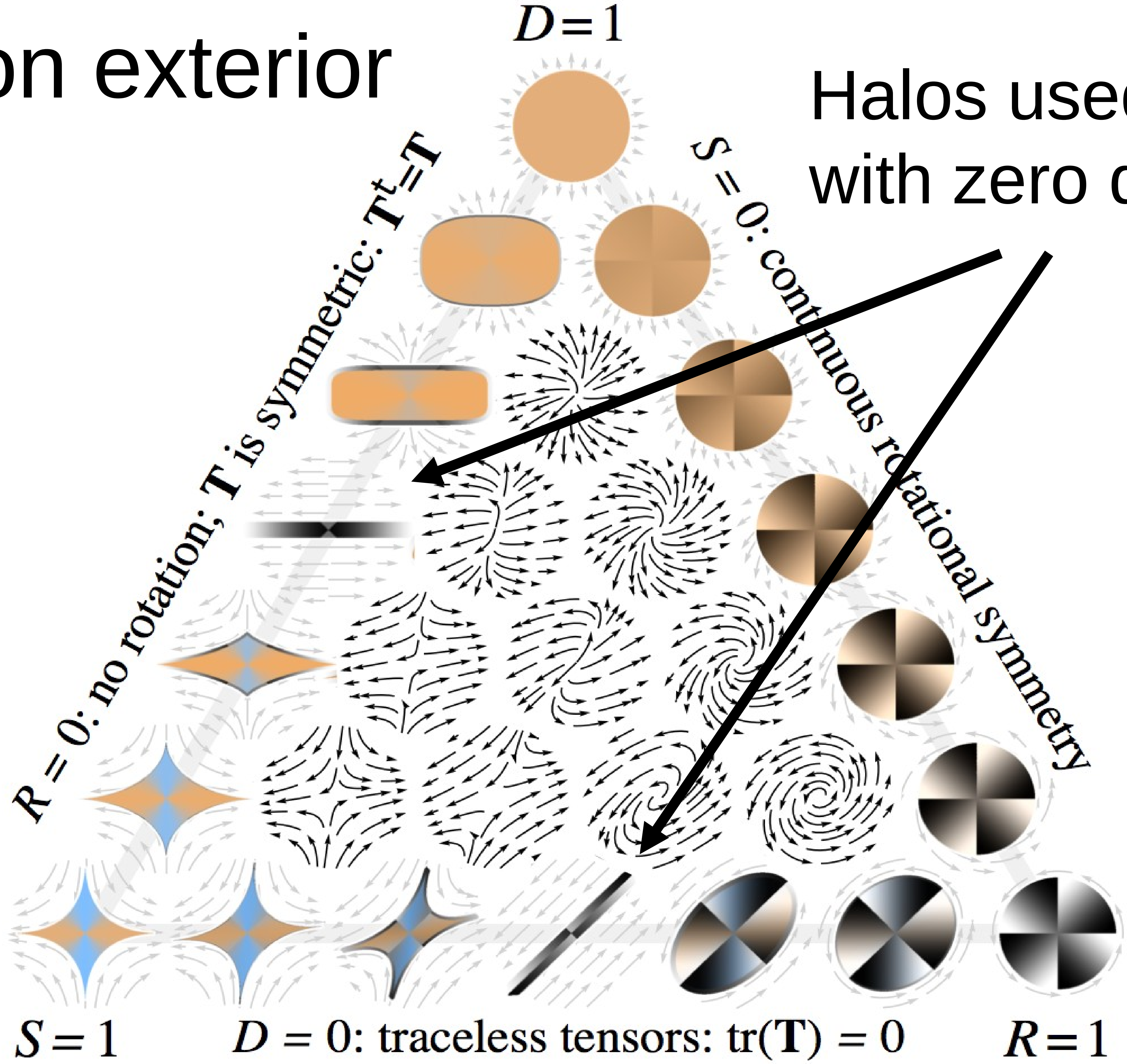
# Rotationally Symmetric Tensors



- Circular shape preserves rotational symmetry
- Luminance gradient
  - Gives rotation direction and magnitude
  - Pattern relates to peripheral drift illusion [Fraser79]
  - Breaks rotational symmetry

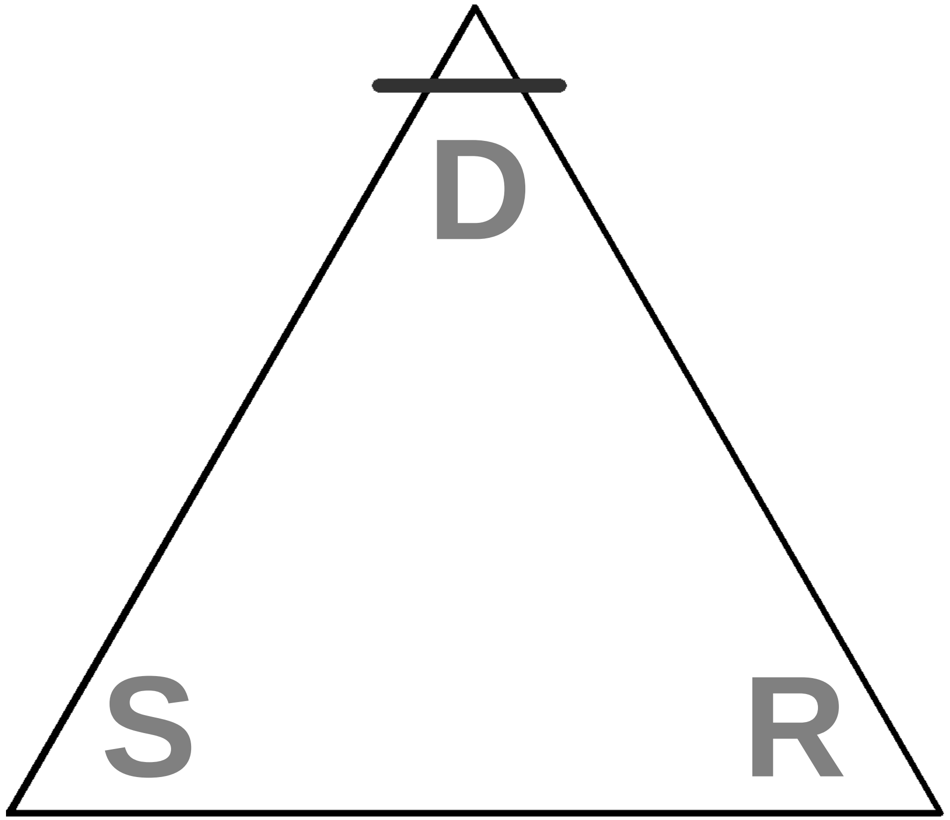
# Glyphs on exterior

Halos used for tensors with zero determinant

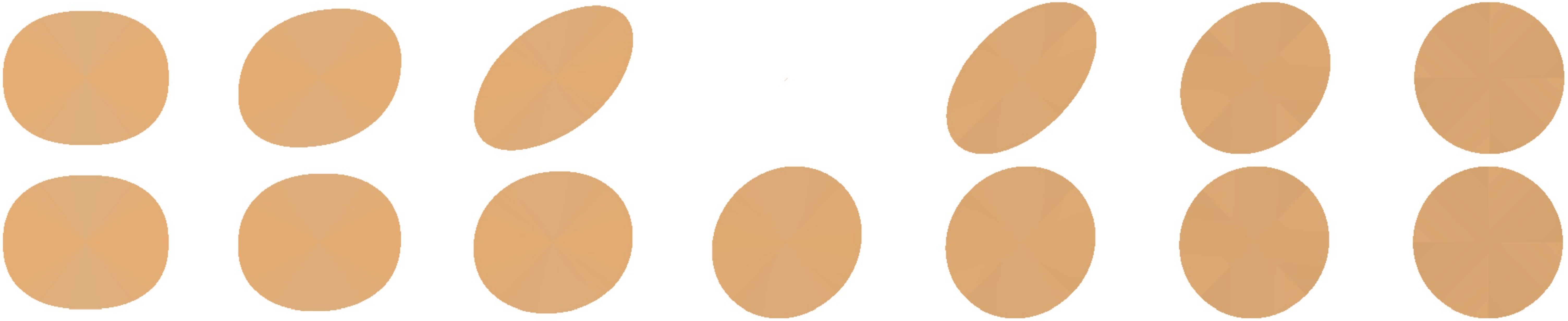


# Parallel eigenvectors

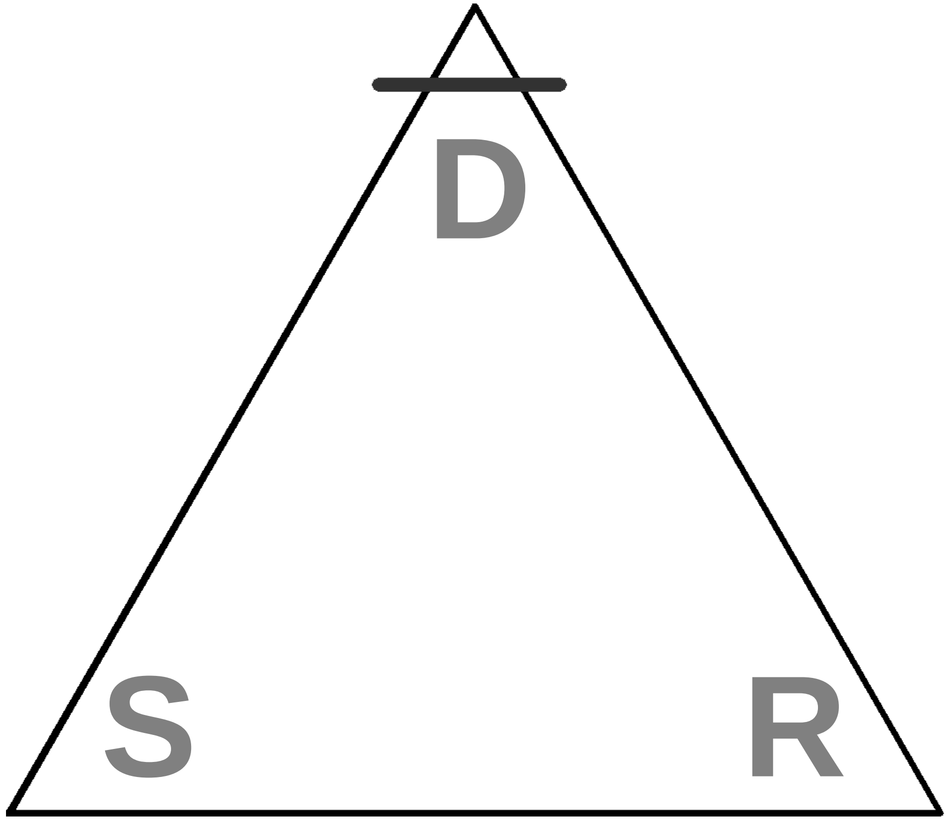
- (pseudo)eigenvectors are parallel when S=R



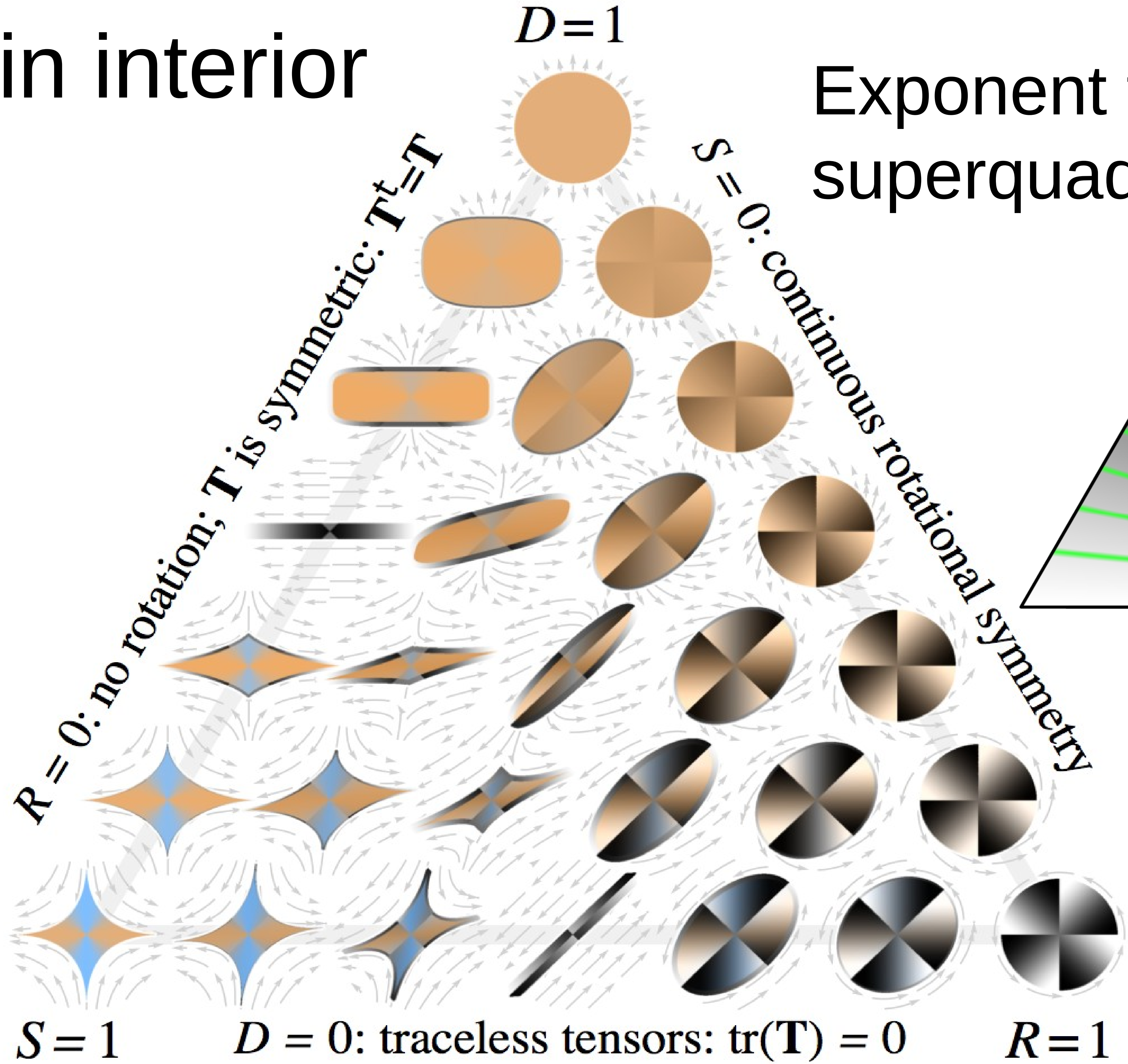
# Parallel eigenvectors



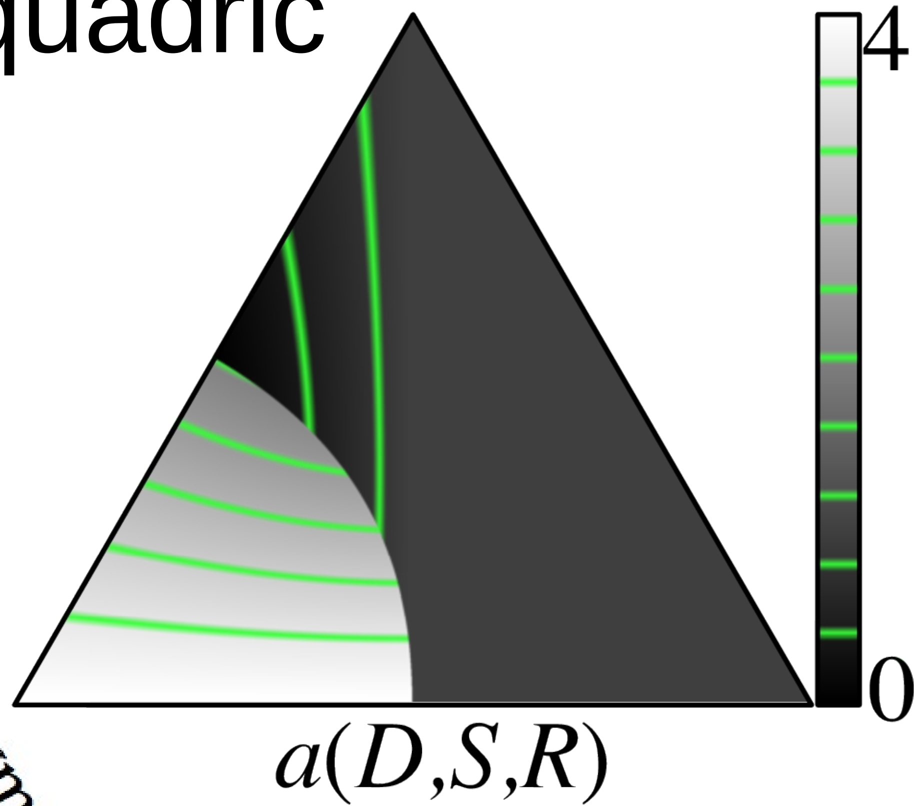
- (pseudo)eigenvectors are parallel when S=R
- Adjust vectors so glyph area indicates tensor determinant (“quasi-eigenvectors”)



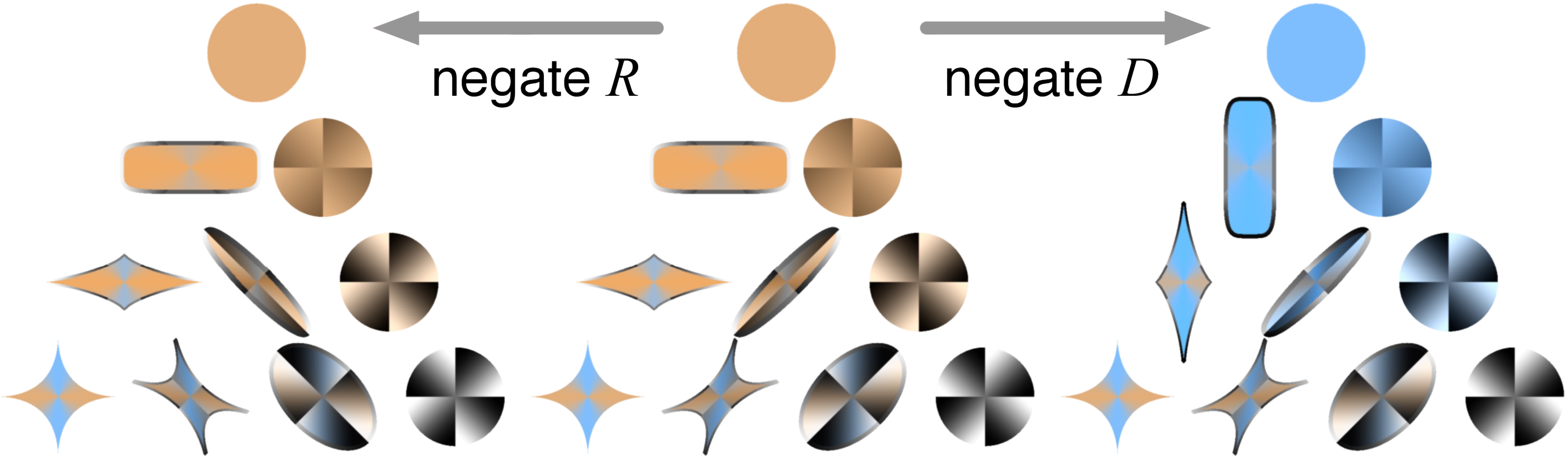
# Glyphs in interior




Exponent for base superquadric



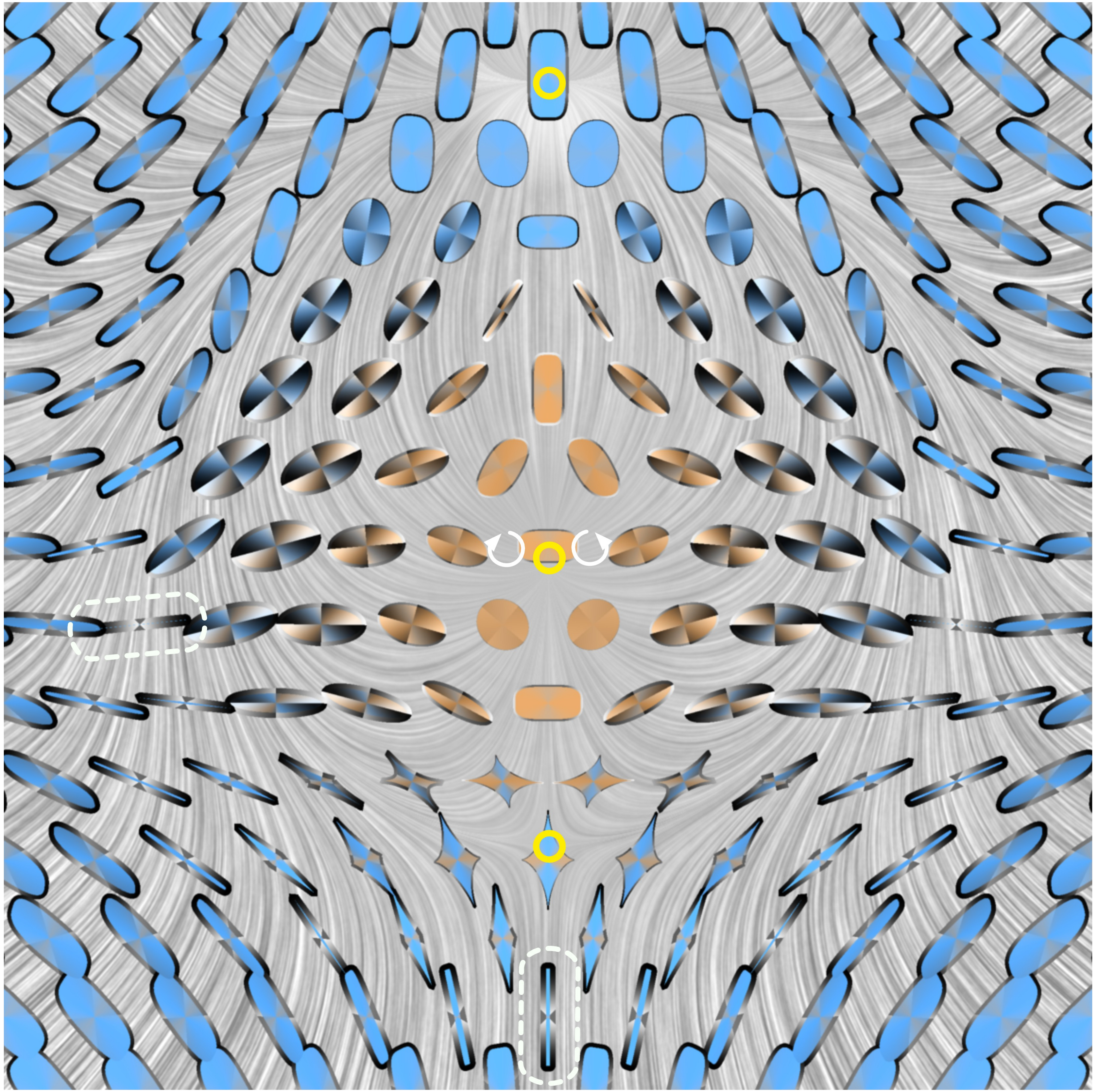
# Negation Legibility




- Negating R is a reversal of rotation direction
- Negating D swaps expansion and contraction

# Results

- Two superimposed Sullivan vortices [Zhang+09]
  - Same magnitude
  - Opposite direction
  - Small horizontal offset
- LIC background



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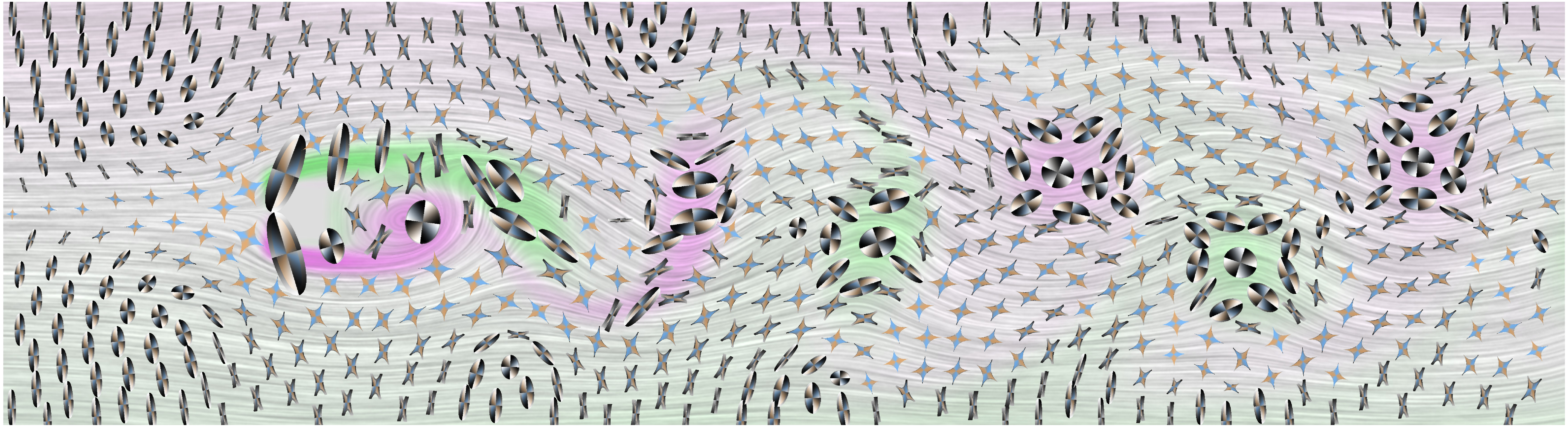

- Central slice of a 3D unsteady flow simulation [ICFDD]
- LIC background
- Magenta and green indicate positive and negative vorticity, respectively

# Conclusion & Future work

- Expanded the class of tensors visible by glyphs
- Constructively guided by Algebraic Vis design
  - Transform legibility: negation, scaling, rotation
  - Continuous, unambiguous
- Captures previous glyphs as special cases
- Designed glyphs within novel D,S,R triangle
- Extension to 3D case
- Evaluation
  - Measure quality of embedding [Demiralp+14]
  - User study in scientific application

# References


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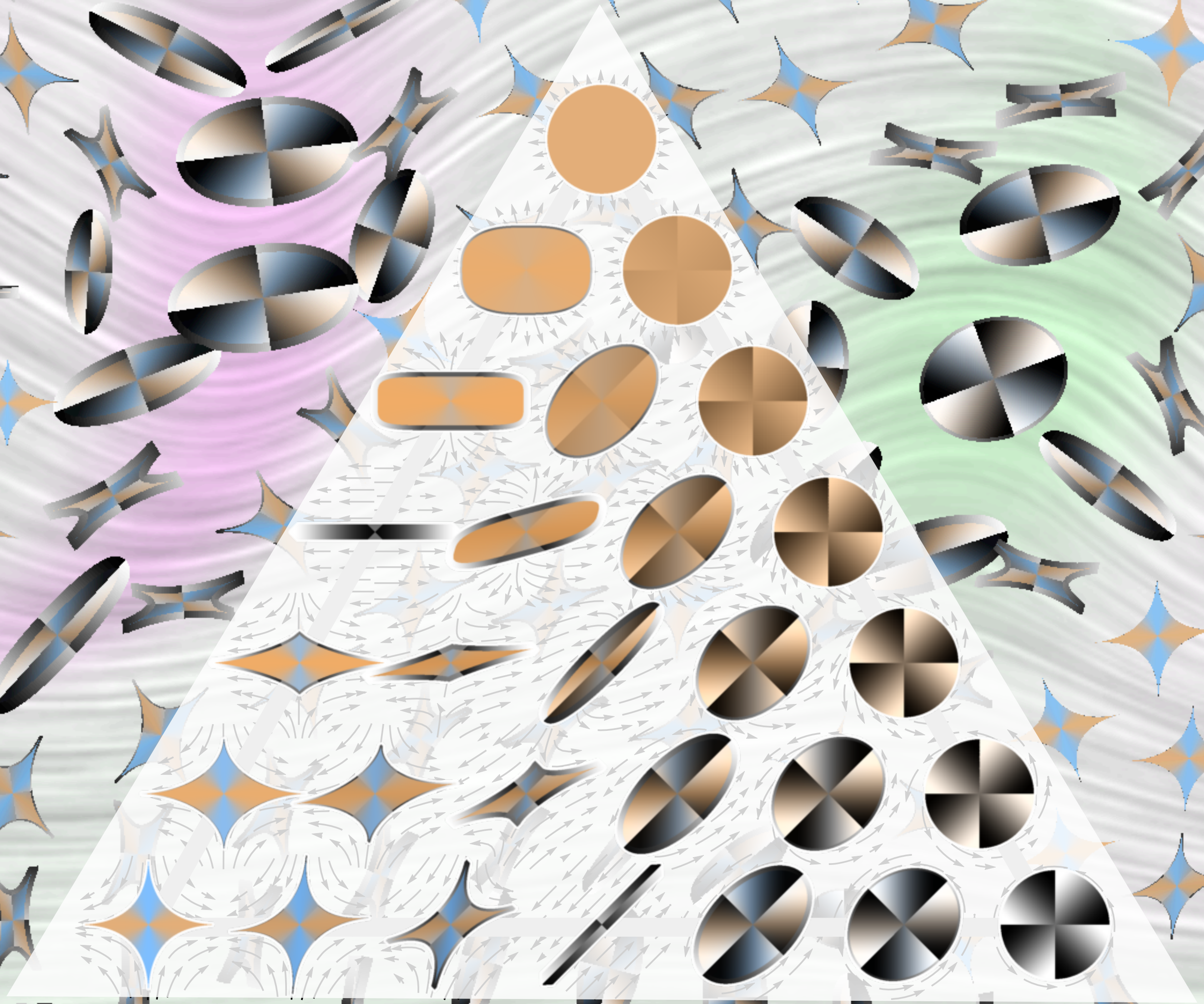

Thank you!
Questions?