Making sense of Math in Vis

Gordon Kindlmann
University of Chicago
glk@uchicago.edu

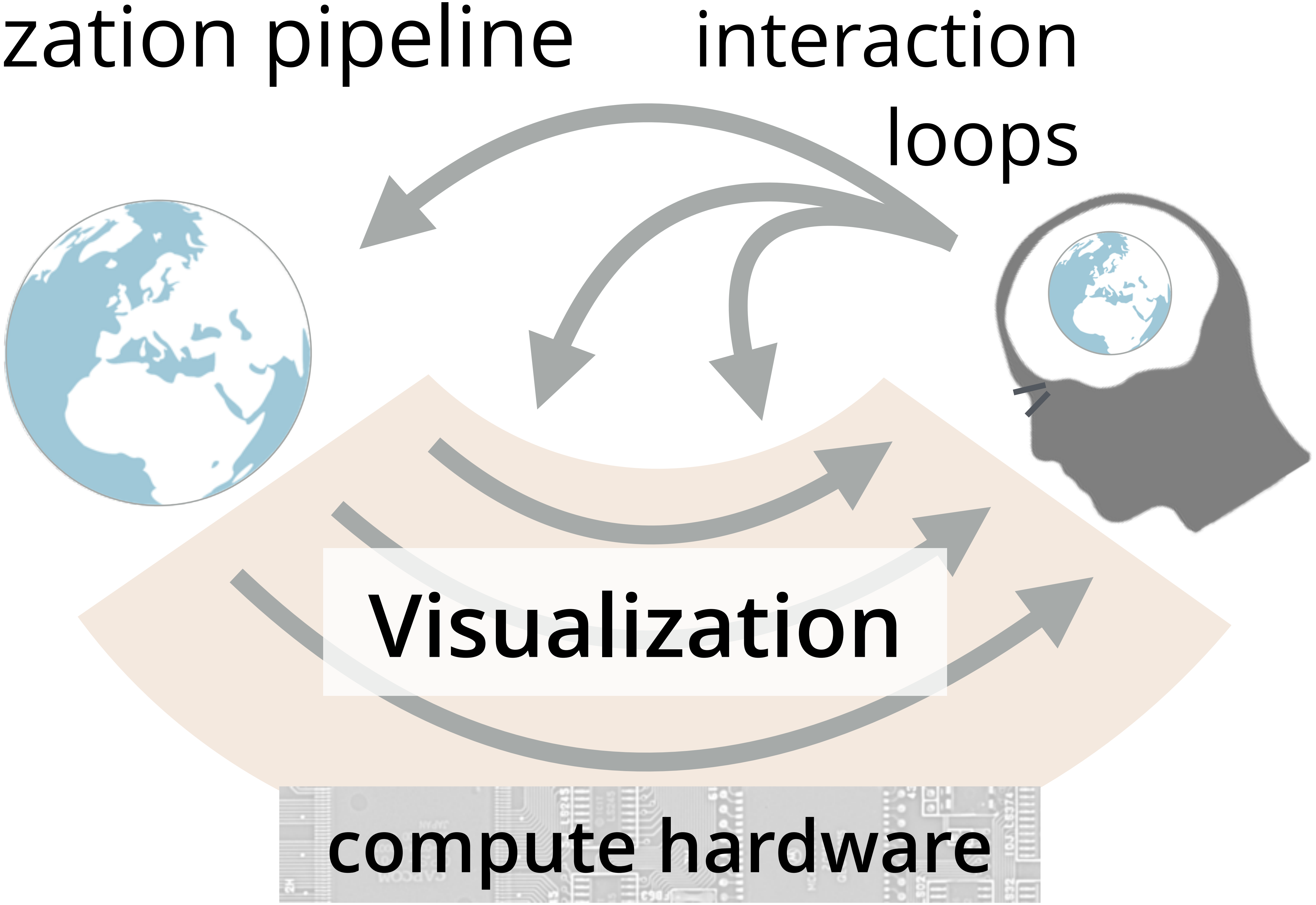
(from seminar description)

http://www.dagstuhl.de/en/program/calendar/semhp/?semnr=18041

Mathematical foundations of visual data analysis. There is a rich tradition of ***mathematical and computational methods used in visualization***, such as topological approaches, feature extraction, numerical sampling and reconstruction methods, numerical integration, differential operators, filtering, dimension reduction, and applications of information theory, partly incorporating uncertainty. While all these methods have a solid mathematical foundation, a careful look at the relation between theories and their role in visual data analysis is needed.

How can it all be organized? What's missing?

Schematic view of Visualization pipeline



Why have Math in Vis?

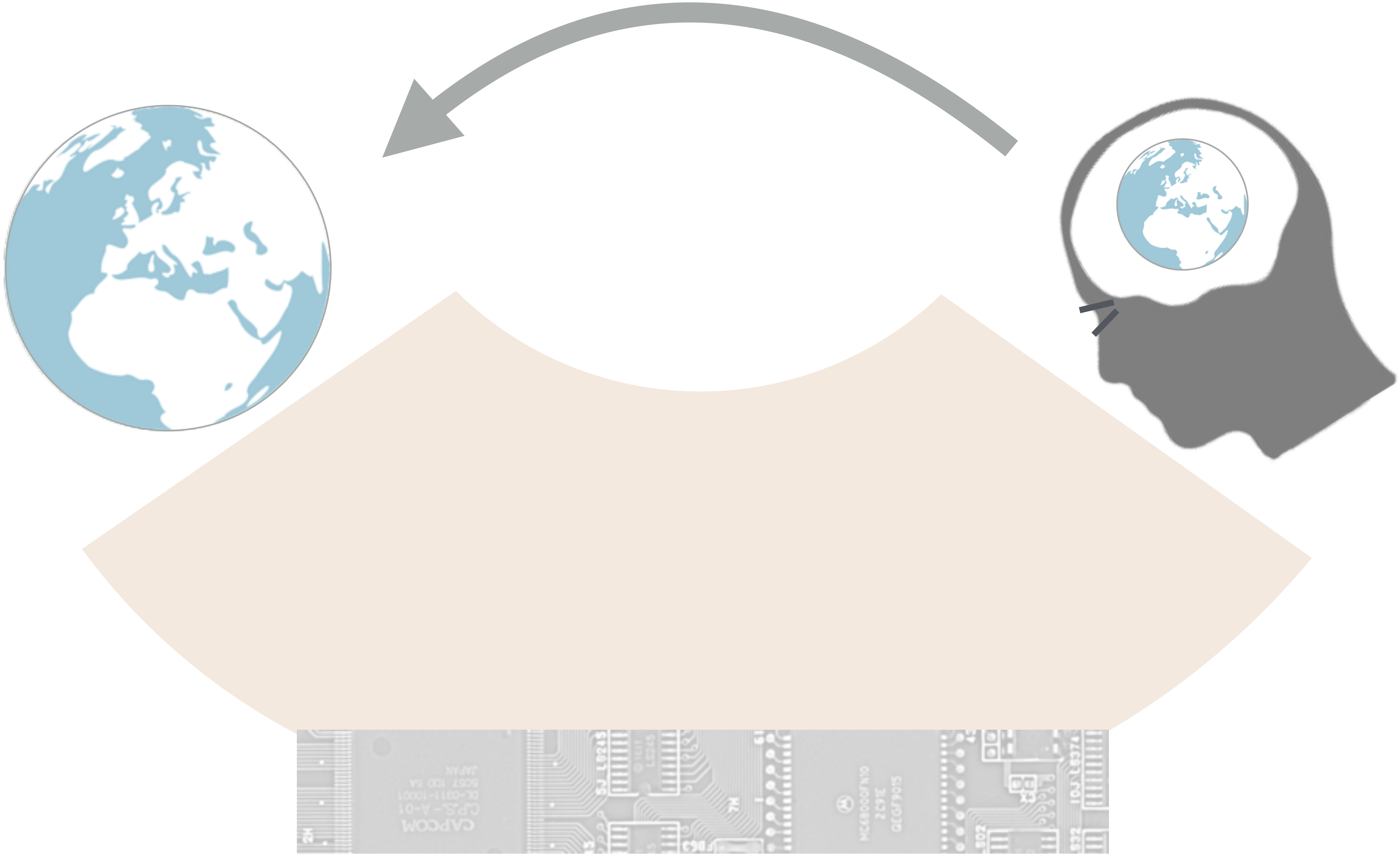
To **Describe**, to Abstract
away specifics, embrace a level of generality
⇒ Usefully **Structure** Visualization Pedagogy

To **Connect**, to Engage
with disciplines that already use math

To **Enrich/Solidify** vis, by leveraging
formalism of mathematics

To Aspire to or Broadcast Sophistication?

Where is Math in Vis?

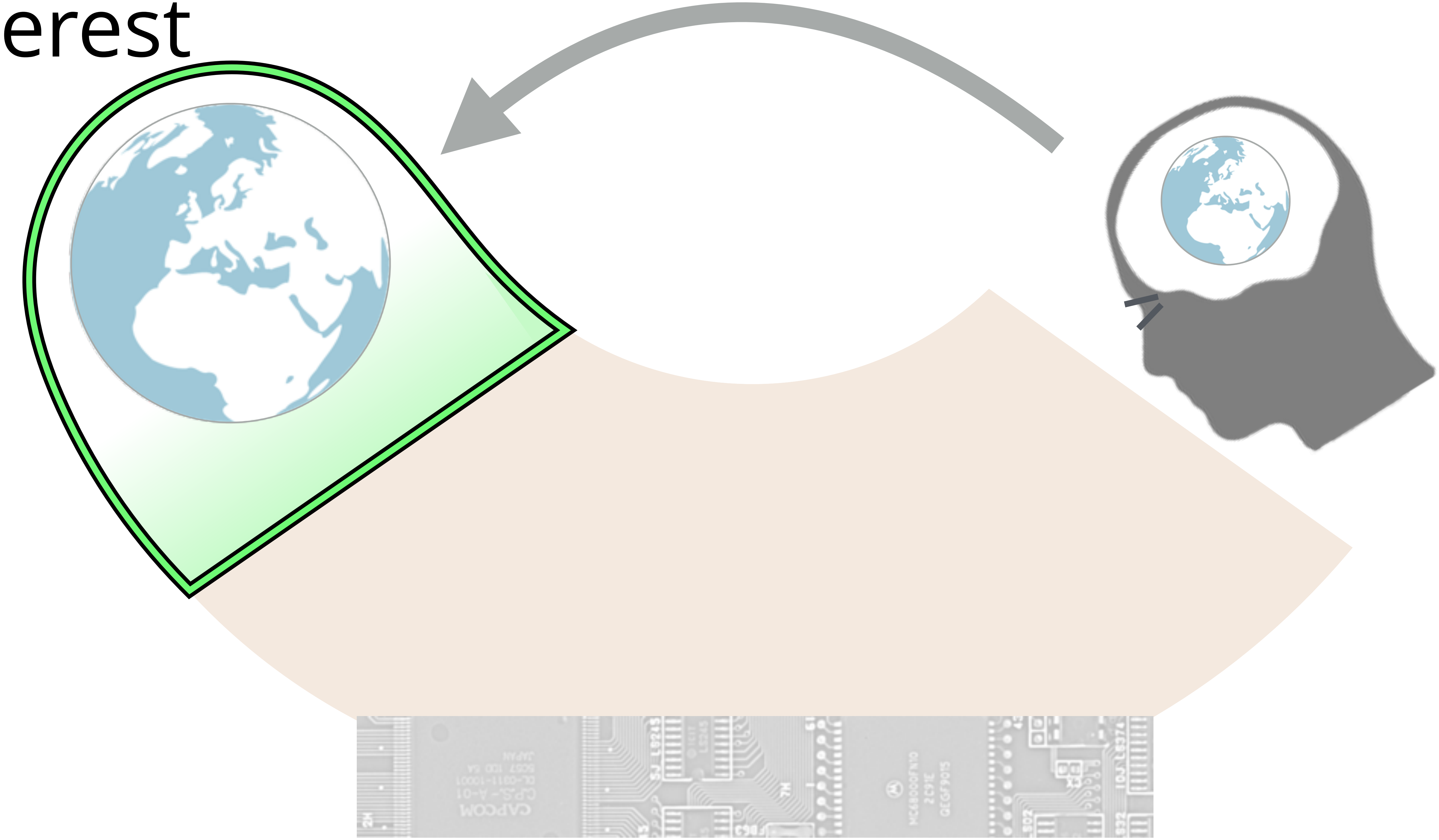


Where is Math in Vis?

In about **6** different places

Help!
correct (now)
&
complete (this week)

(1) in **models** of objects/phenomena of interest



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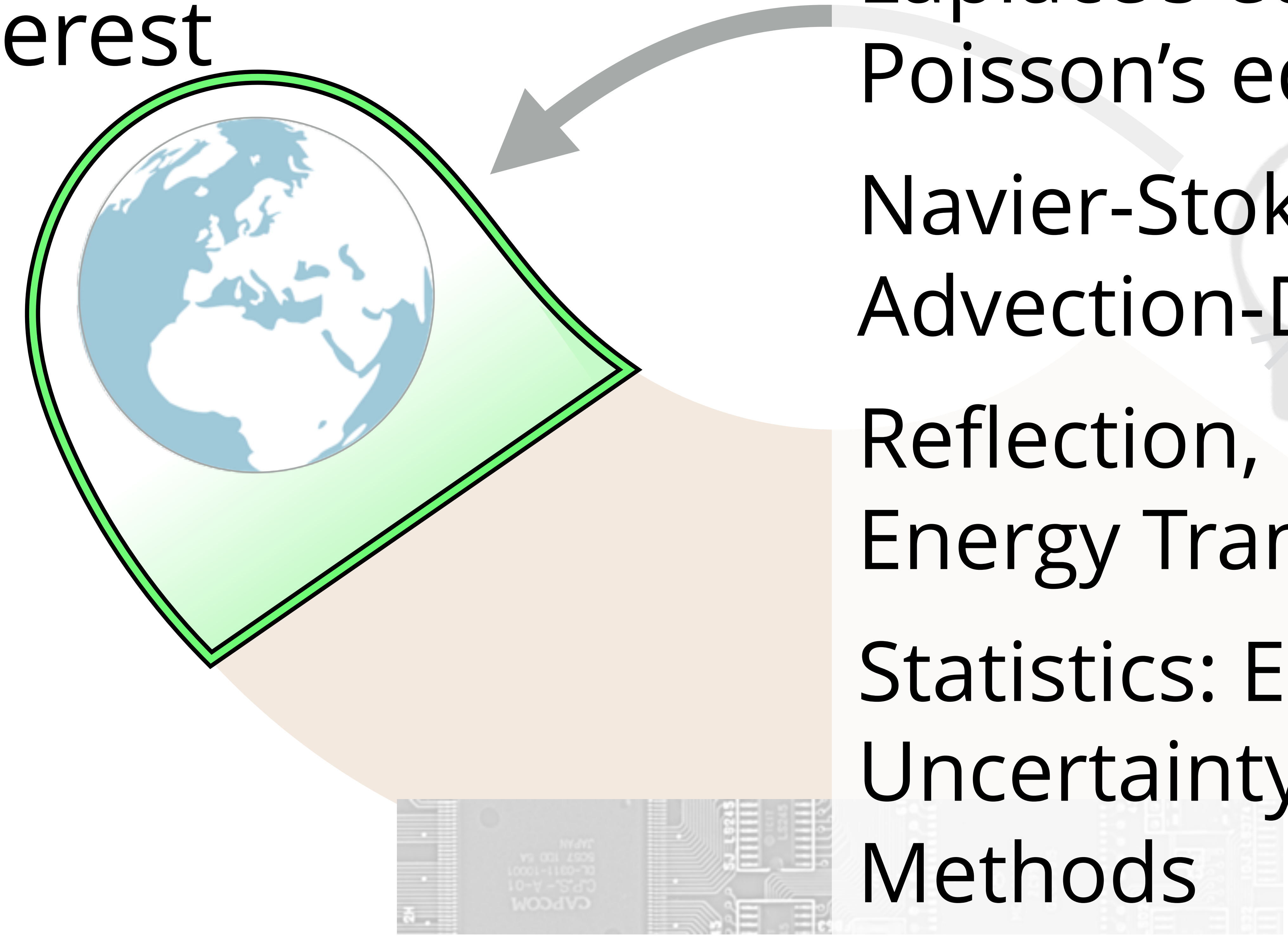


(scientific computing)

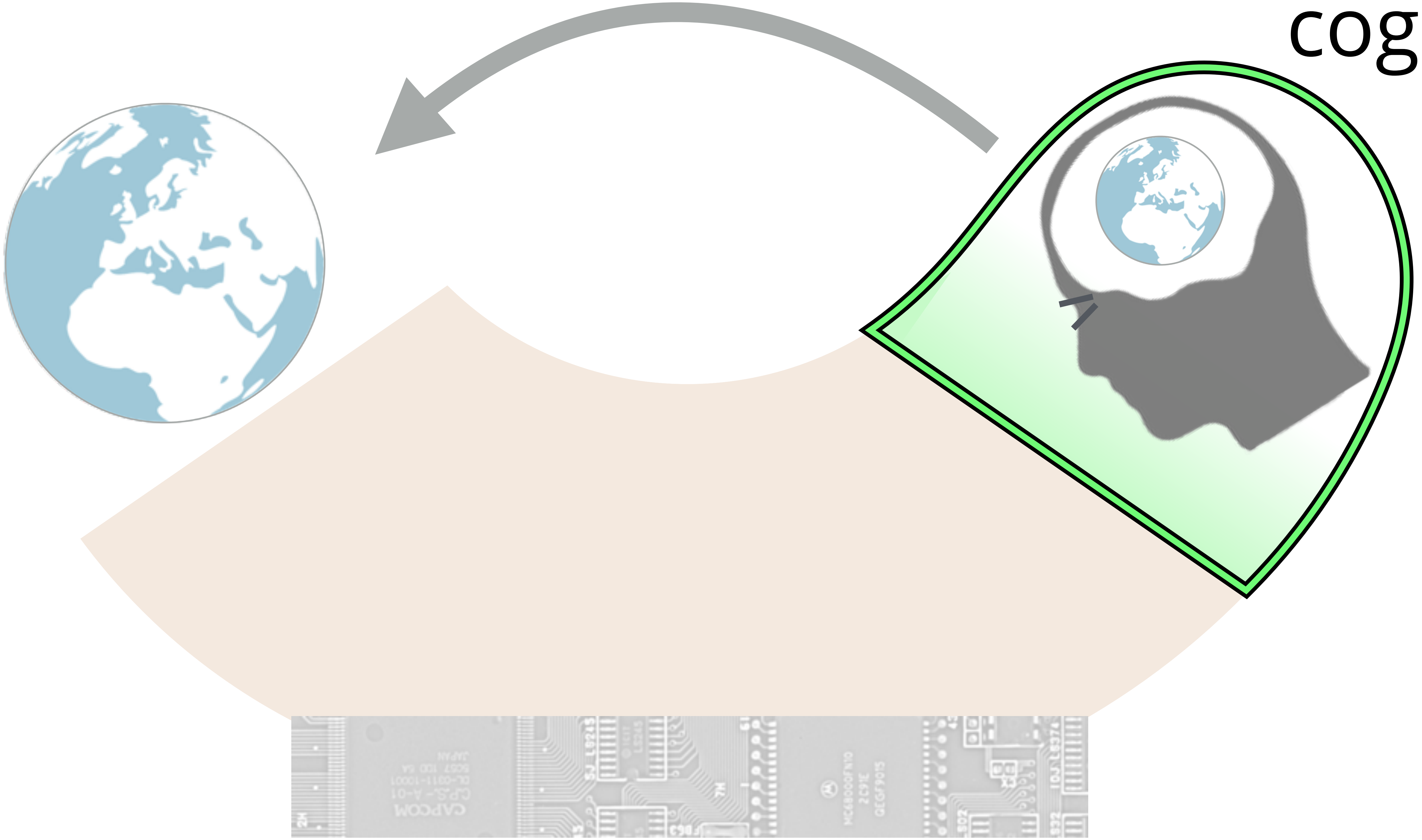
Laplace's equation,
Poisson's equation

Navier-Stokes, Heat,
Advection-Diffusion PDEs

Reflection, Illumination,
Energy Transport

Statistics: Ensembles,
Uncertainty, Bayesian
Methods

(2) in **models** of human perception and cognition



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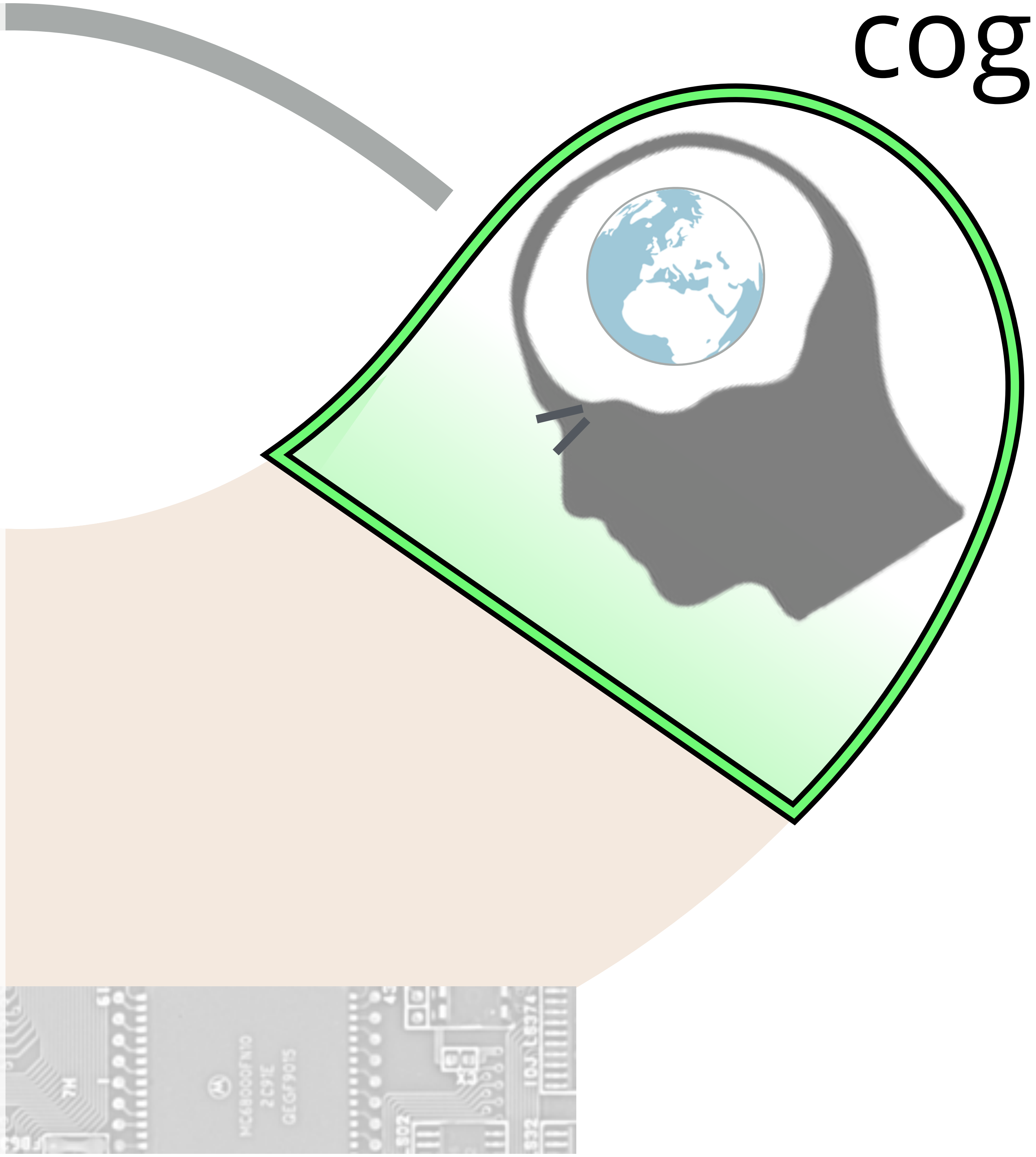


(Perceptual Psychology)

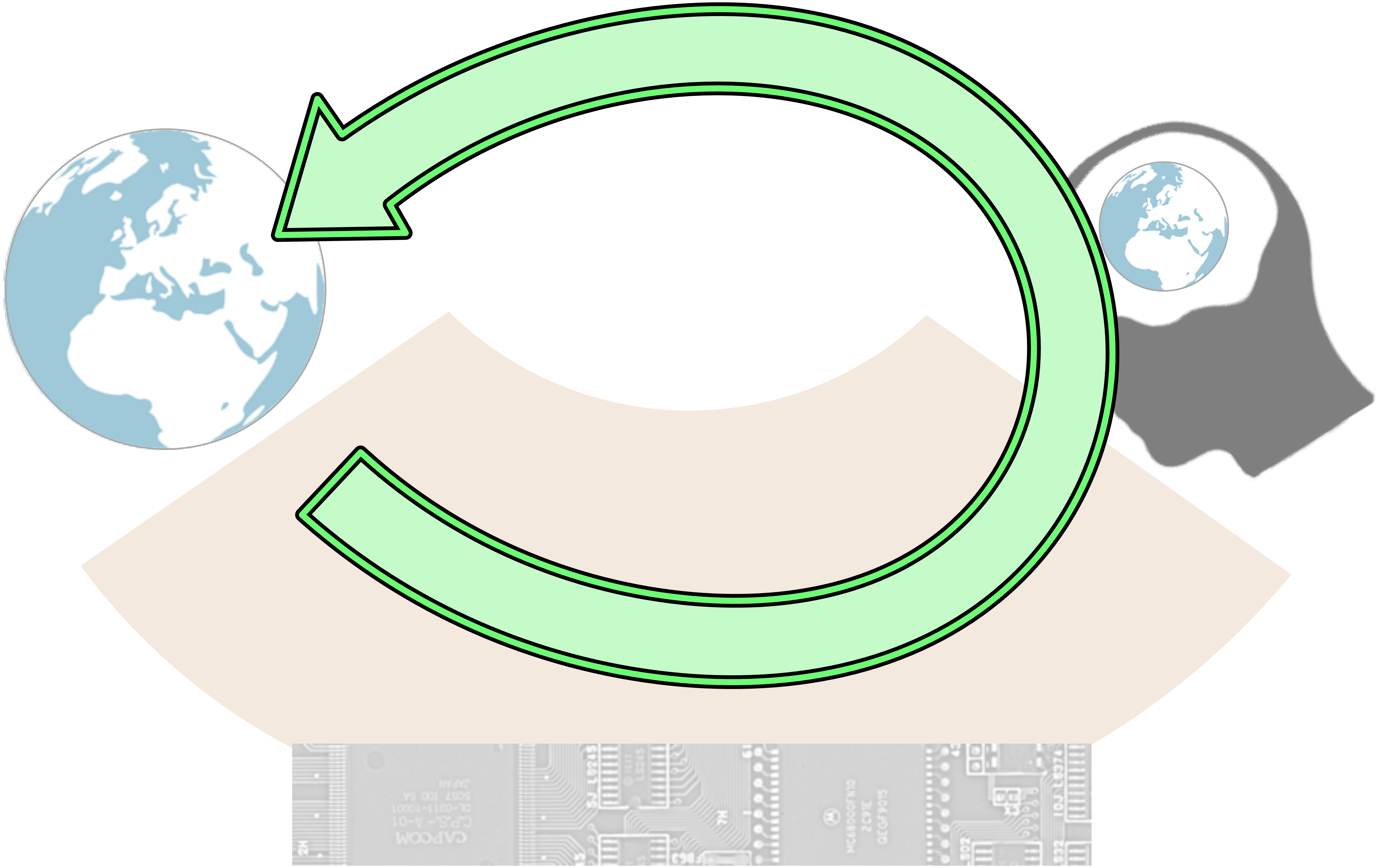
Stevens Law, Weber-Fechner Law

Opponent Color Channels, Color Appearance Models (e.g. CIECAM02)

Gabor Wavelets (perception of scale)

Bayesian Models Of Gestalt [Jäkel-QuantitativeGestalt-VR-2016]

(3) in **empirical** study of how people perform some task using vis



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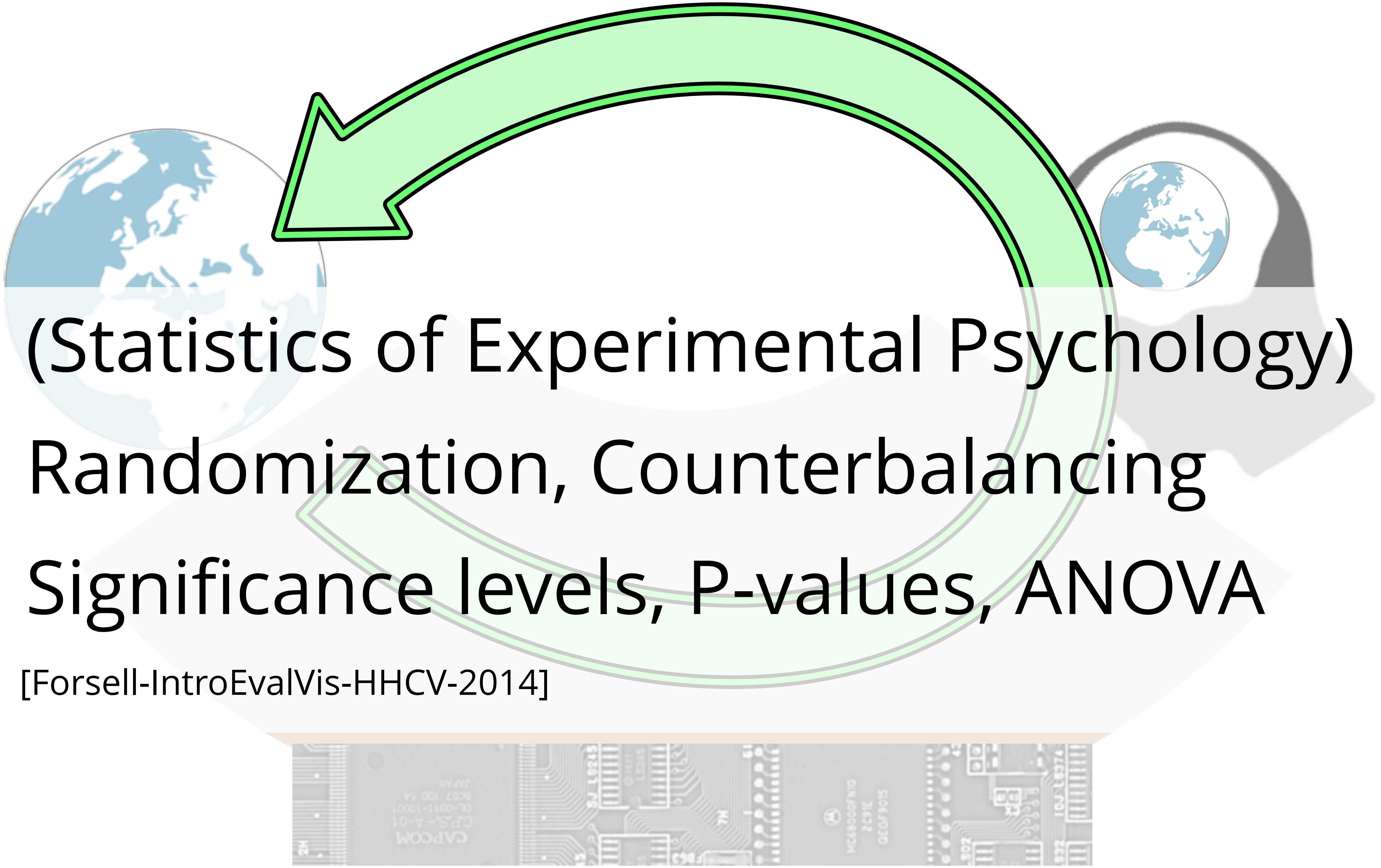


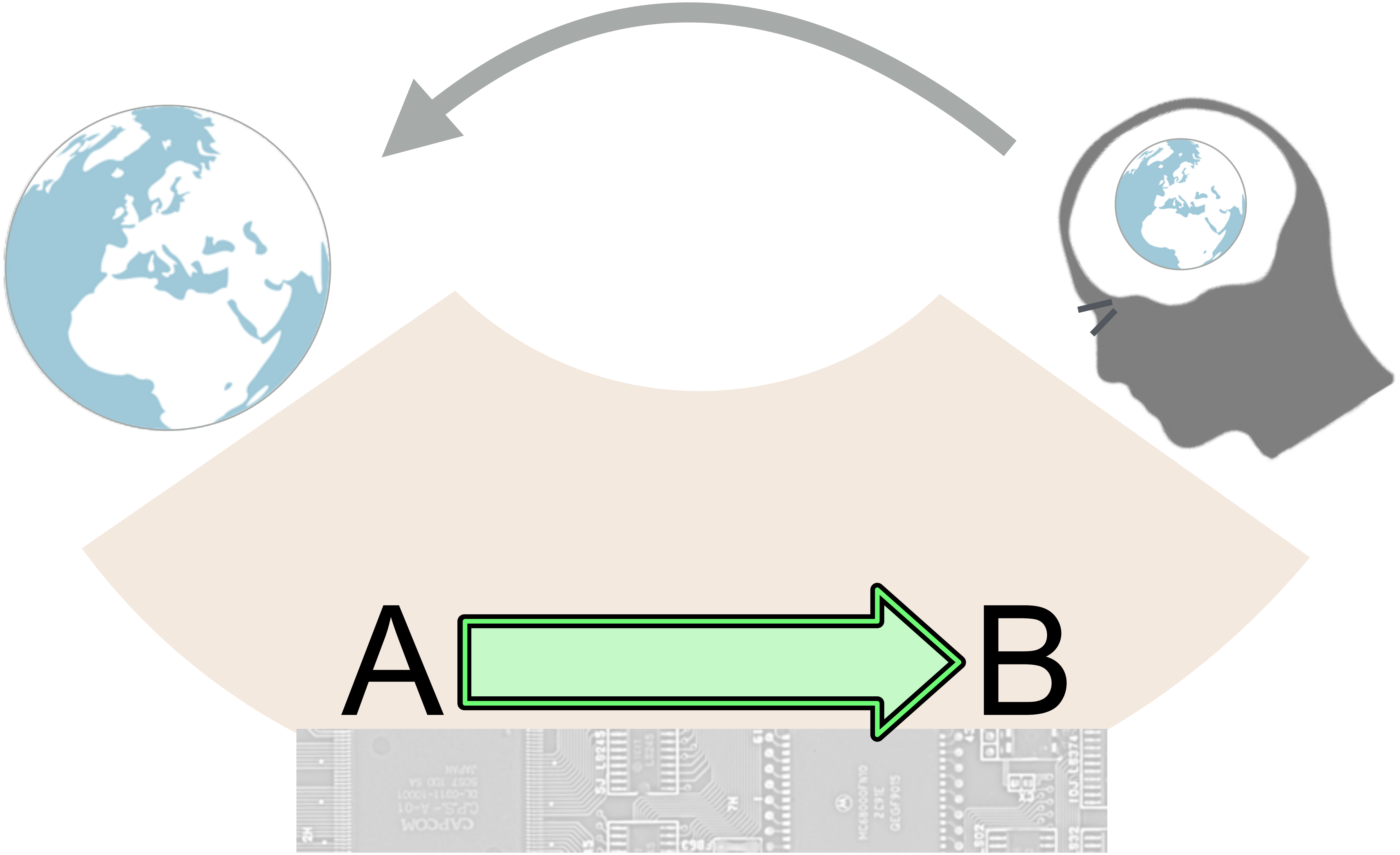
(Statistics of Experimental Psychology)

Randomization, Counterbalancing

Significance levels, P-values, ANOVA

[Forsell-IntroEvalVis-HHCV-2014]

(4) in **definitions** of essential overall **goal** of vis method



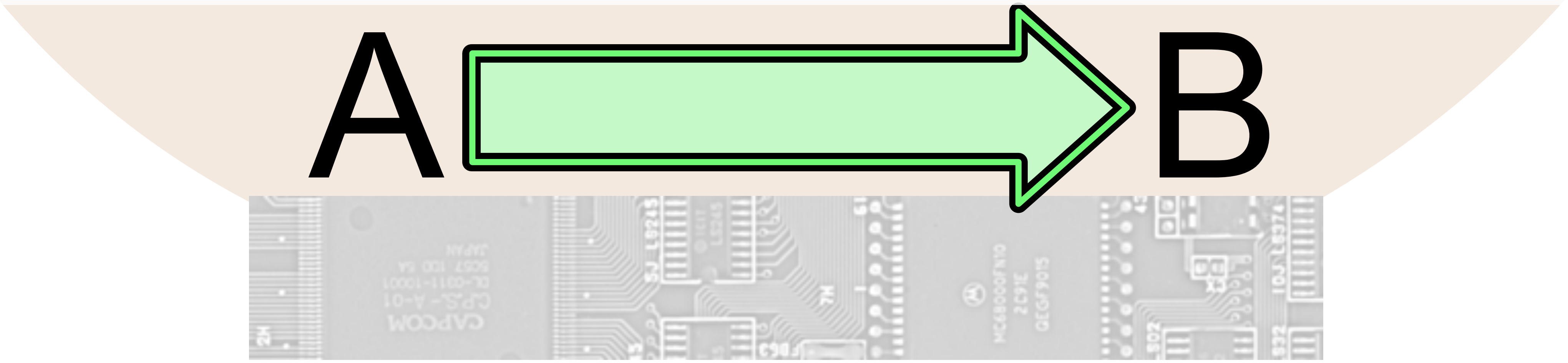
(4) in **definitions** of essential overall **goal** of vis method

Feature Extraction: Isocontours, Parallel Vectors Operator, Ridges and Valleys, Vortex Cores

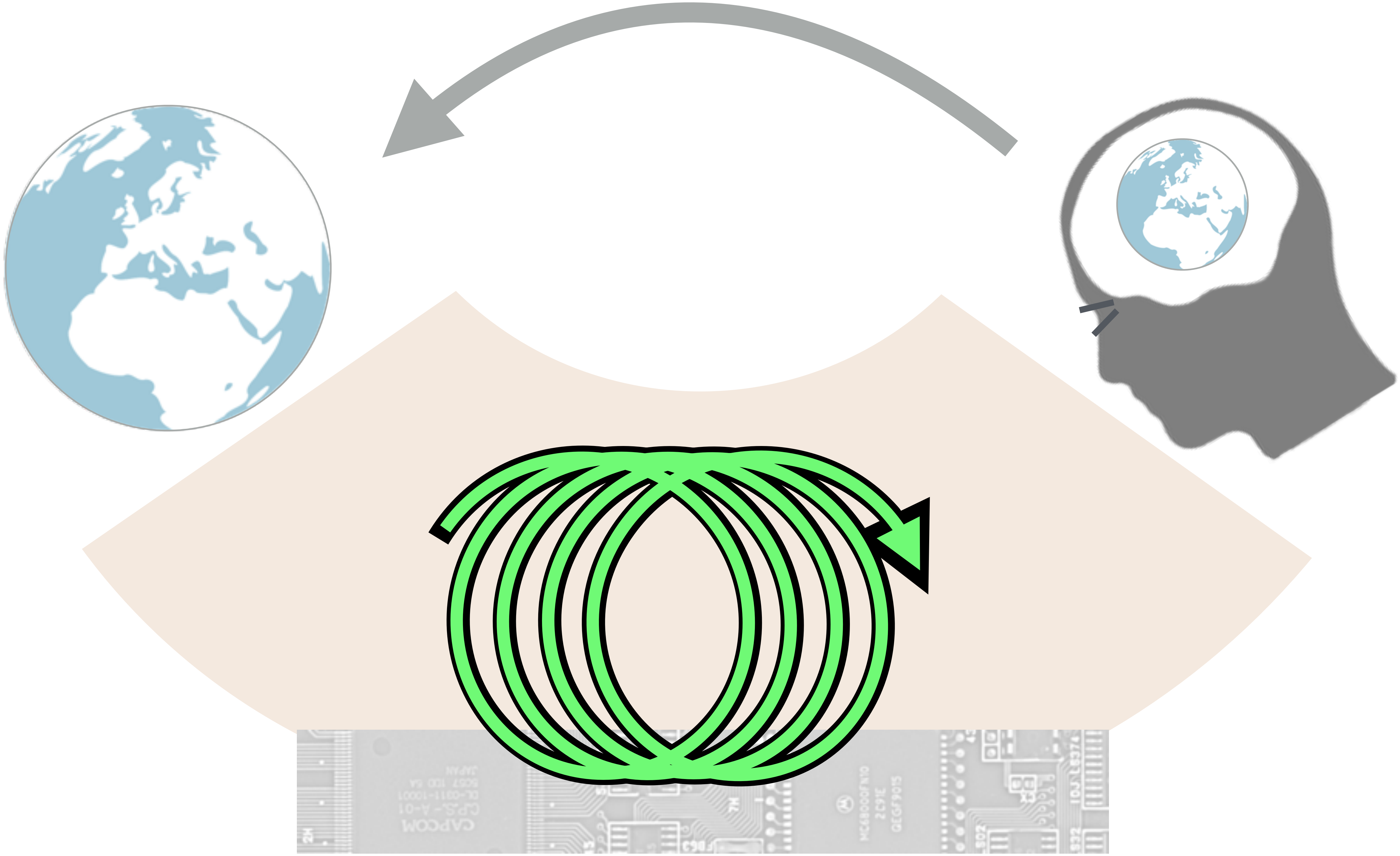
Topological methods: Morse-Smale Complex, Reeb Graph

Points & Graphs: Principal Component Analysis, Spectral Clustering, Dimensionality Reduction, Graph Drawing

Solving PDEs with Finite Elements: Dirichlet/Neumann boundary conditions, Galerkin Method, Linear/Spectral Elements



(5) to **implement** low-level parts of overall method

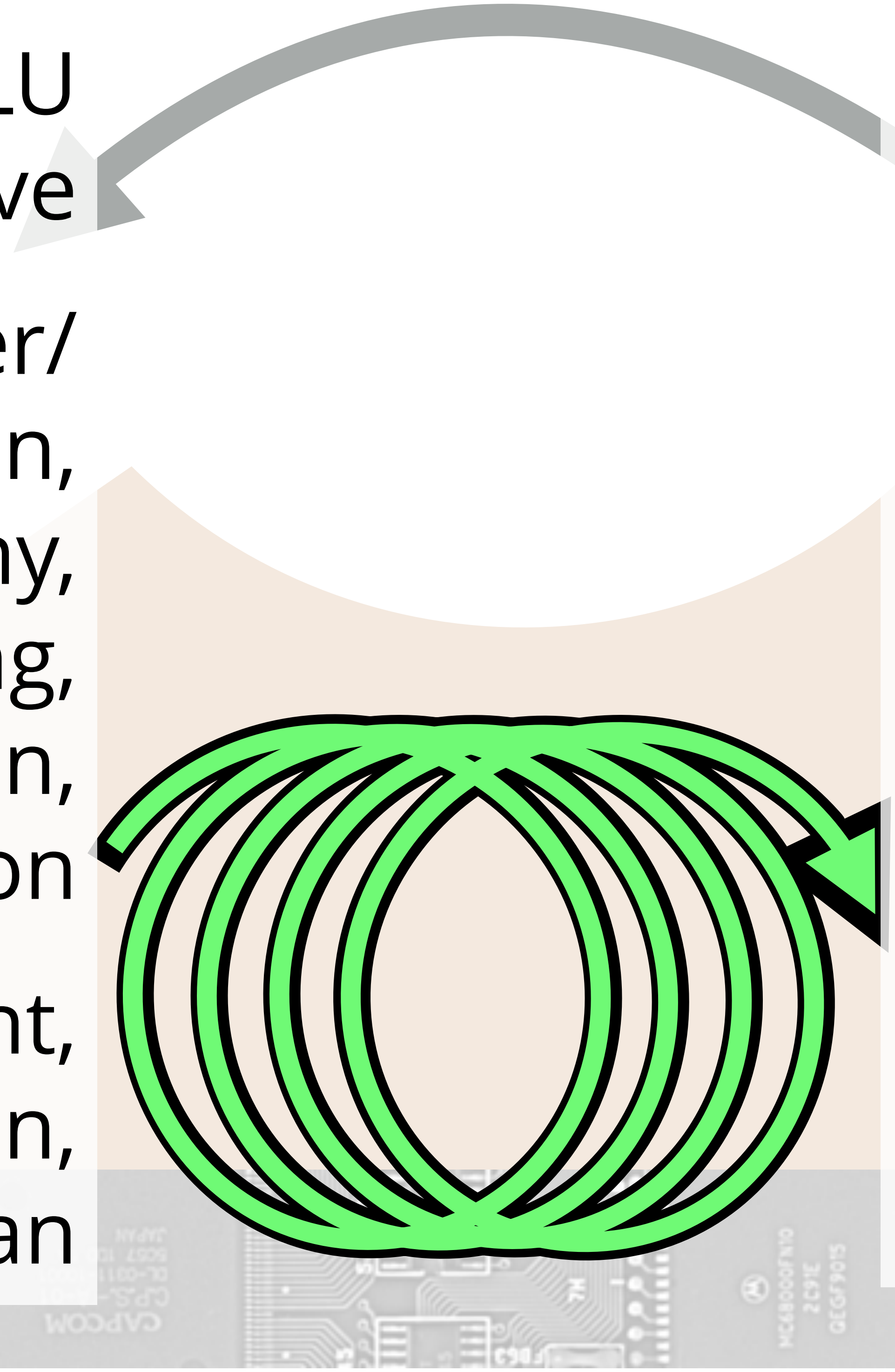


(5) to **implement** low-level parts of overall method

Linear algebra: LU decomposition, eigensolve

Numeric Methods: Euler/ Runge-Kutta Integration, Streamlines, Tractography, Newton root finding, Newton optimization, Kahan summation

Derivatives: Gradient, Jacobian, Laplacian, Hessian

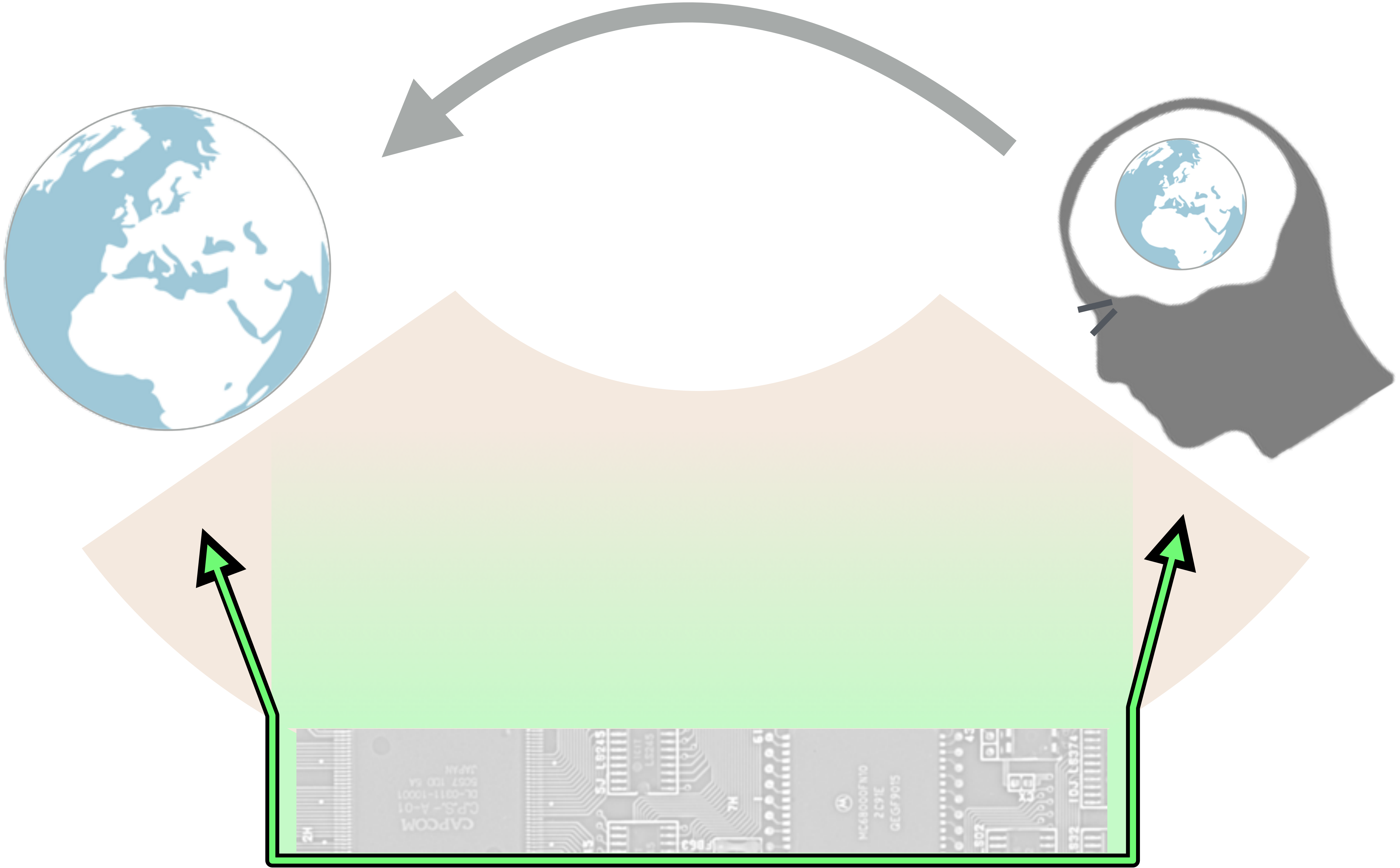


Computing w/ Approximations: Taylor Series, Fourier Series, Wavelets

Signal processing: Nyquist Sampling, Reconstruction by Convolution

Sequence Data: matching, searching, Smith–Waterman alignment

(6) to **characterize** performance of compute hardware for task



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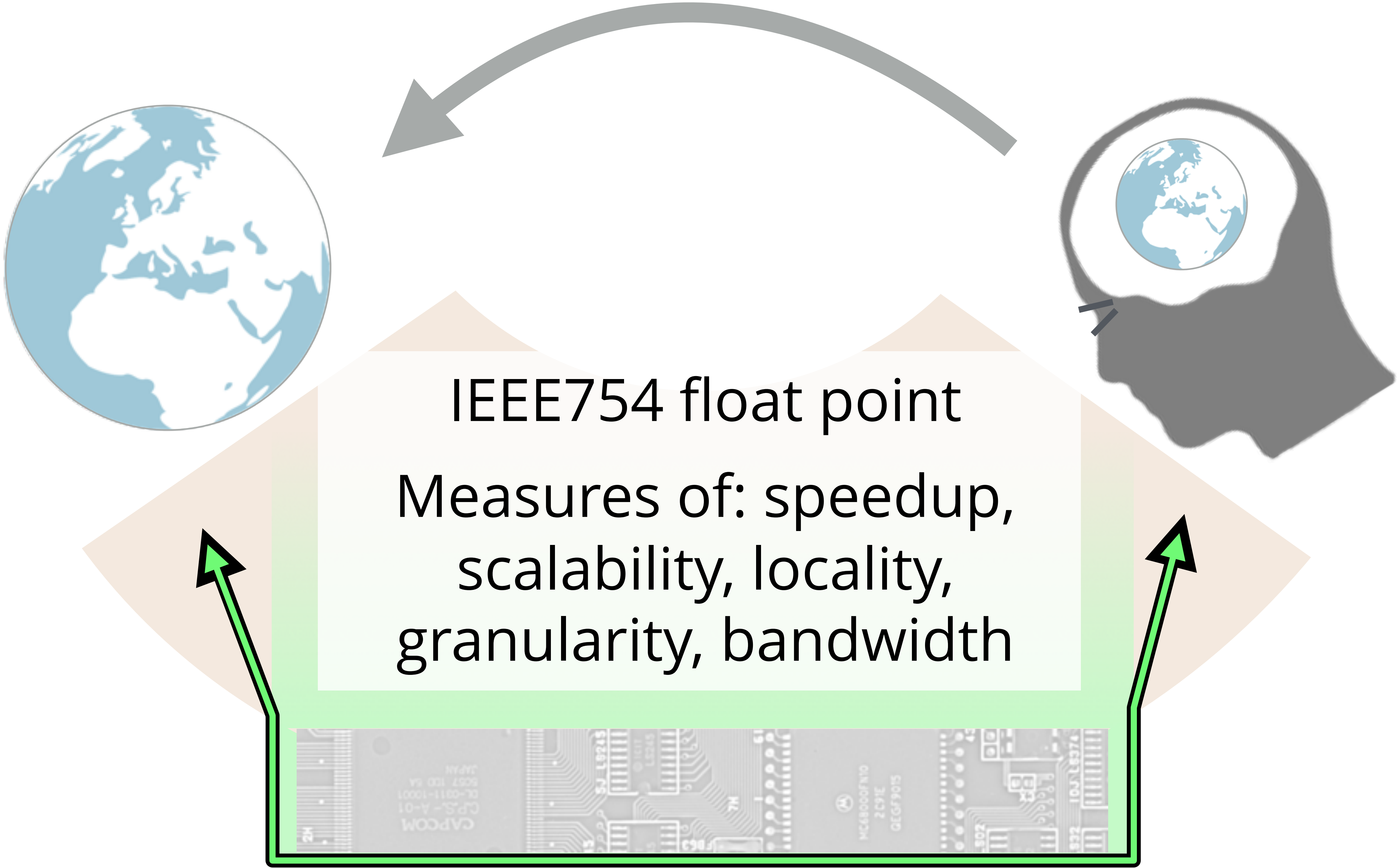
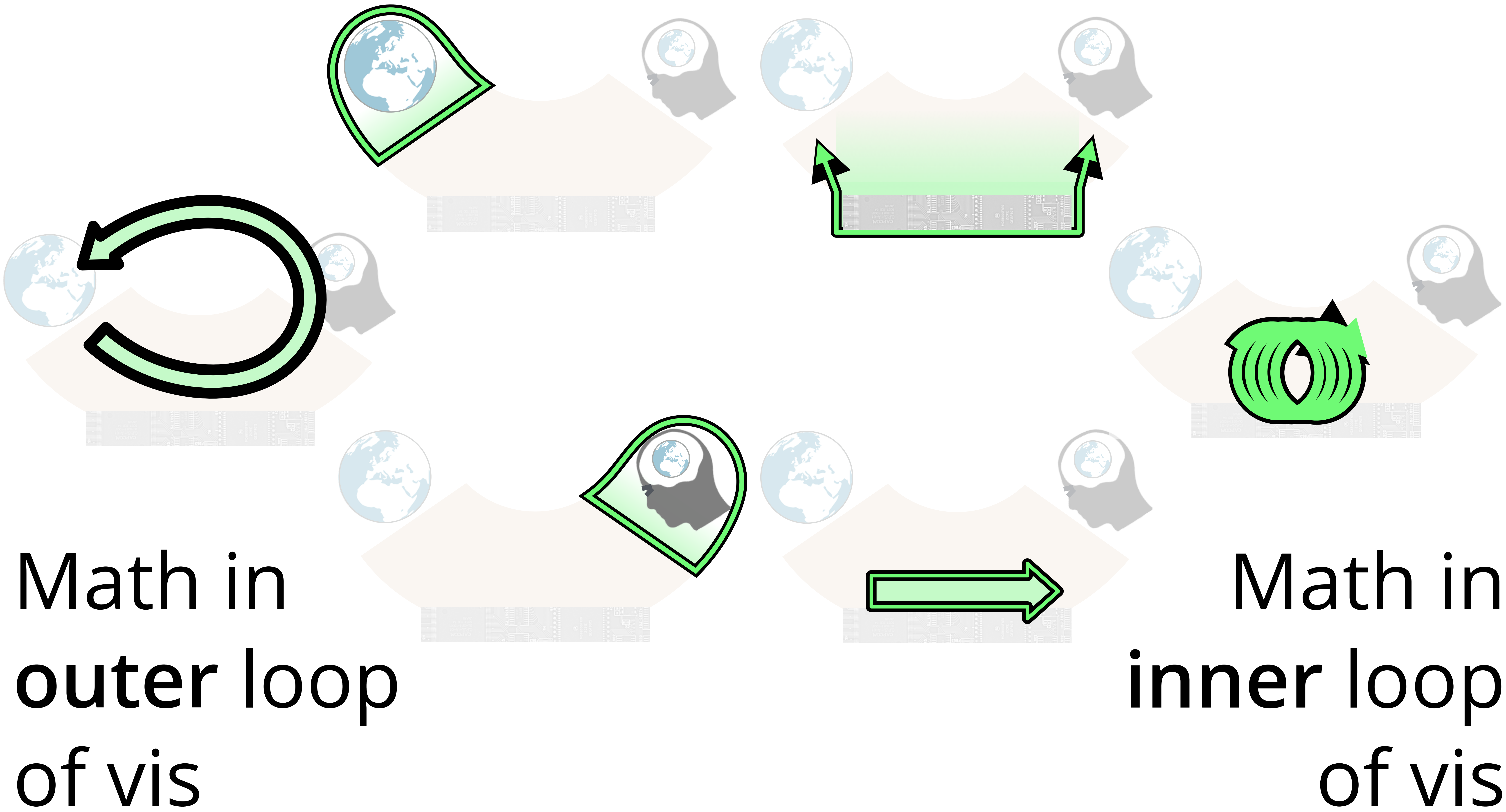


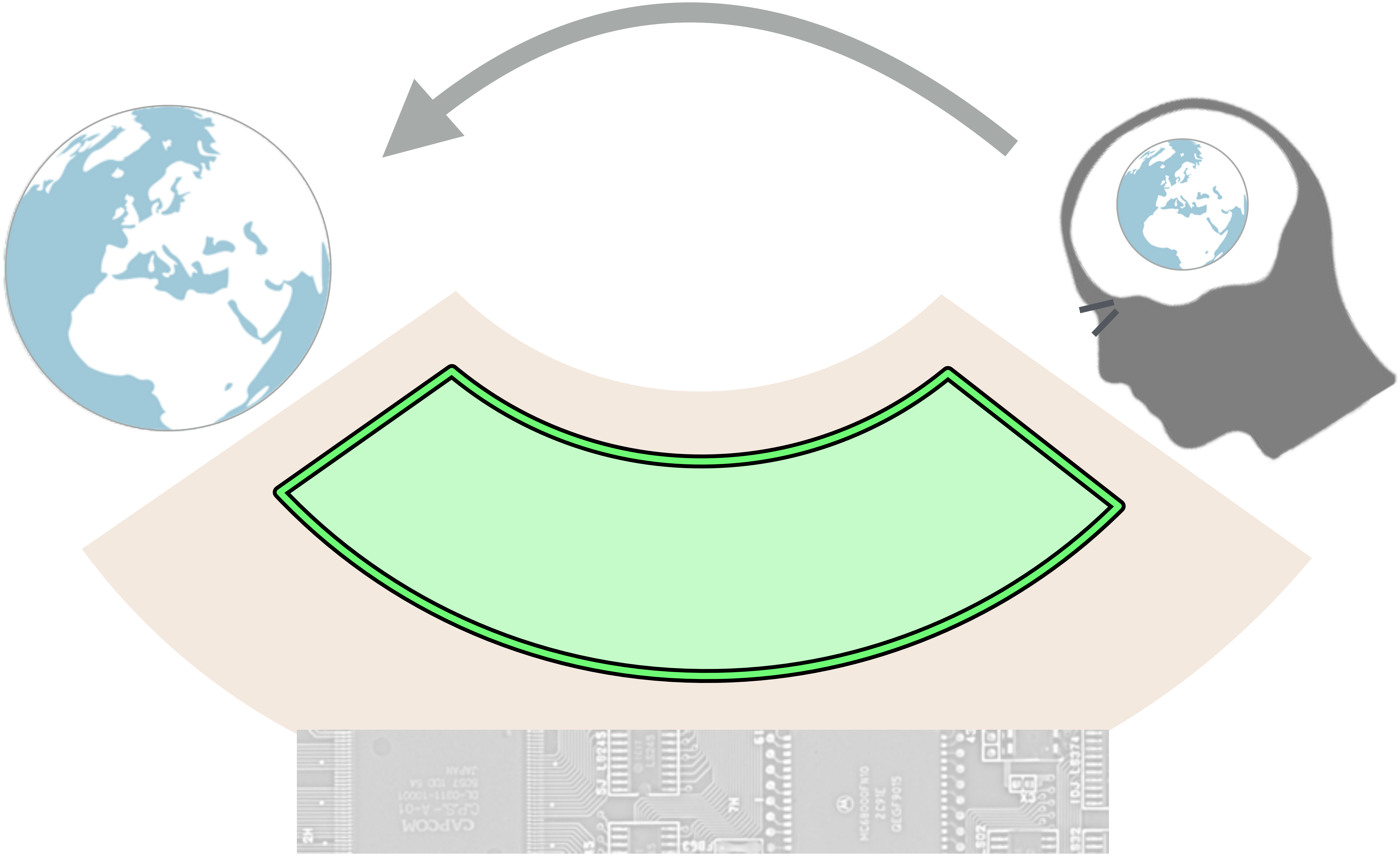

Math in **outer** loop of vis
Math in **inner** loop of vis

What's missing?

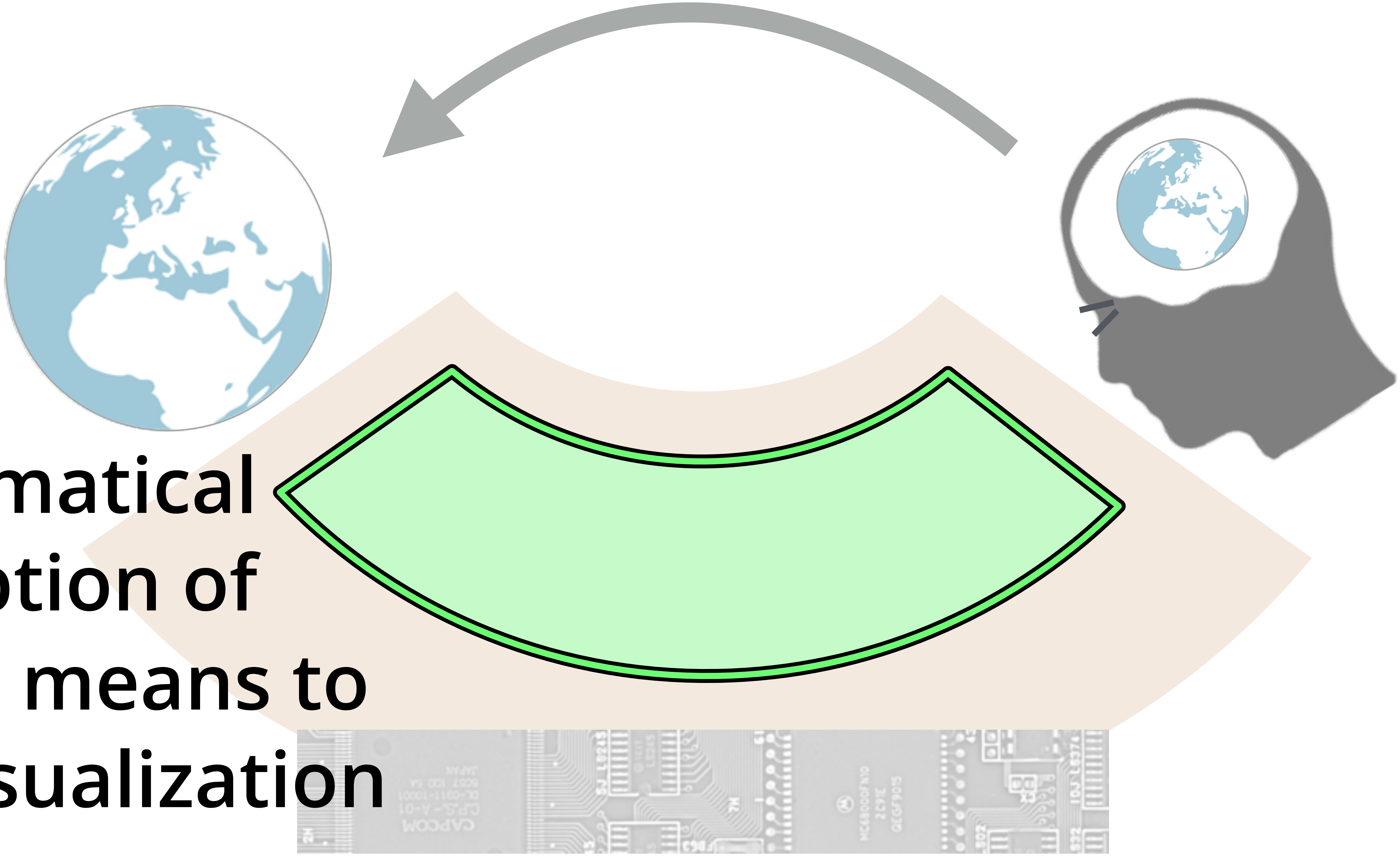
Math in **outer** loop of vis

Math in **inner** loop of vis

Math **of** vis (not **in** vis)



Math **of** vis (not **in** vis)



Mathematical description of what it means to be a visualization

William Hibbard [Hibbard-StructuresOfData-DIDV-1995]

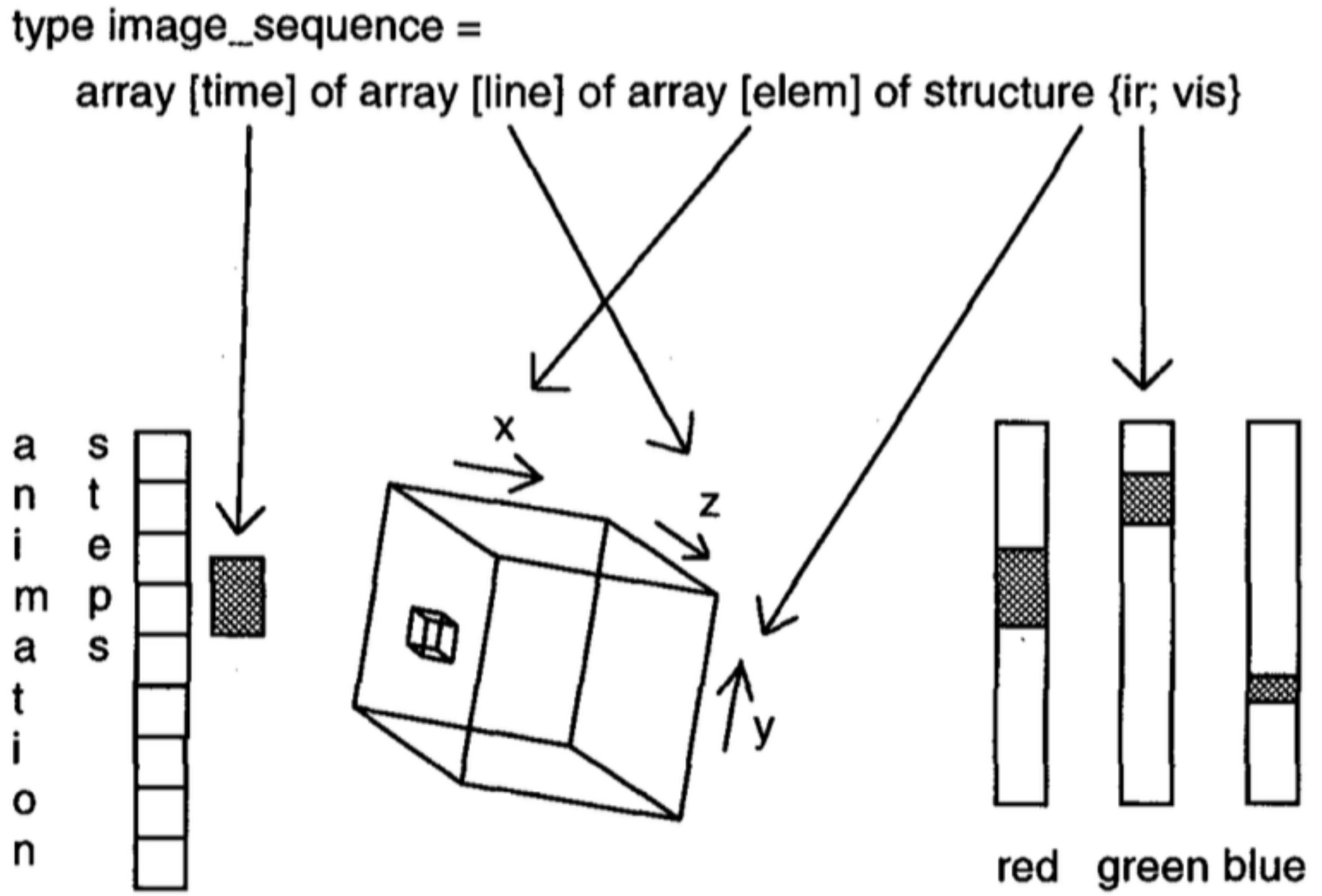
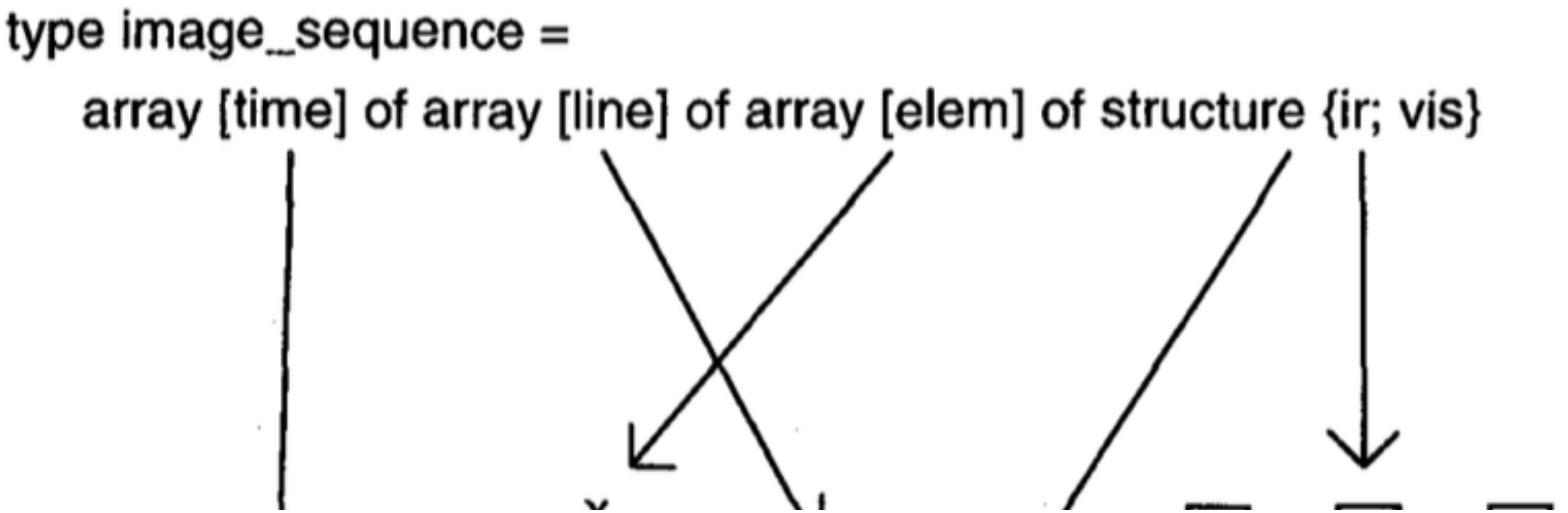


Figure 3. A mapping from a data aggregate to a display aggregate is decomposed into mappings from data primitives to display primitives.

William Hibbard [Hibbard-StructuresOfData-DIDV-1995]

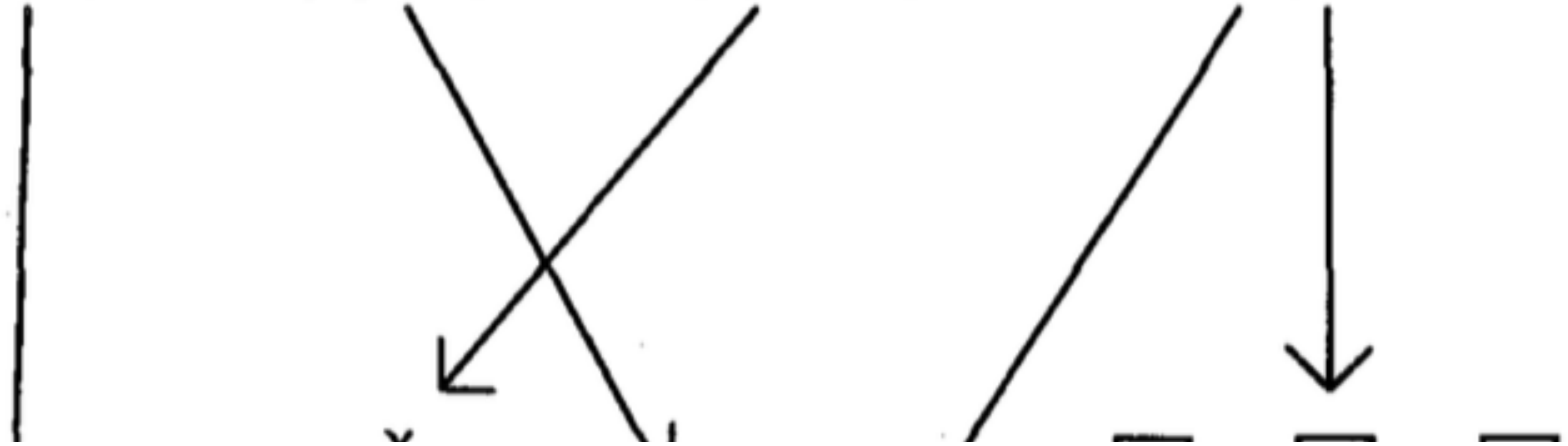


We have briefly investigated how mathematical structures on data can be used to define conditions on the visualization mapping from data to displays. The first three conditions that we discussed are that $D : U \rightarrow V$ map:

The algebraic structure of U to the algebraic structure of V (i.e., D is linear).
The metric structure of U to the metric structure of V (i.e., D is isometric).
The lattice structure of U to the lattice structure of V (i.e., D is a lattice isomorphism).

William Hibbard [Hibbard-StructuresOfData-DIDV-1995]

type image_sequence =
array [time] of array [line] of array [elem] of structure {ir; vis}



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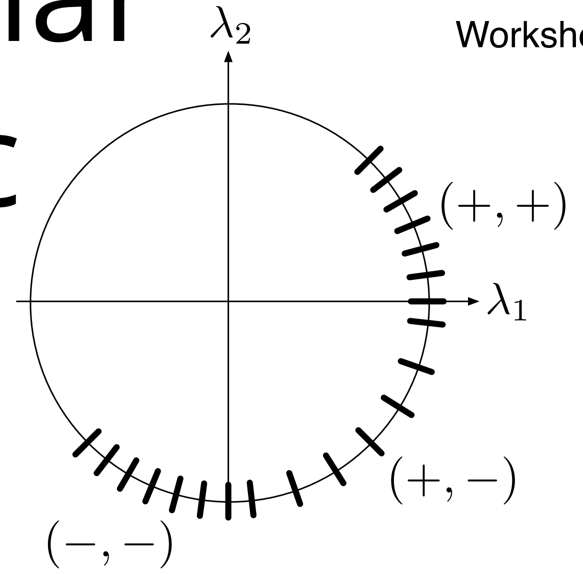
[Demiralp-VisualEmbedding-CGnA-2014]

Dagstuhl Seminar 09251 Scientific Visualization (July 2009)

Worksheet for design of 2D tensor glyphs   Gordon L. Kindlmann



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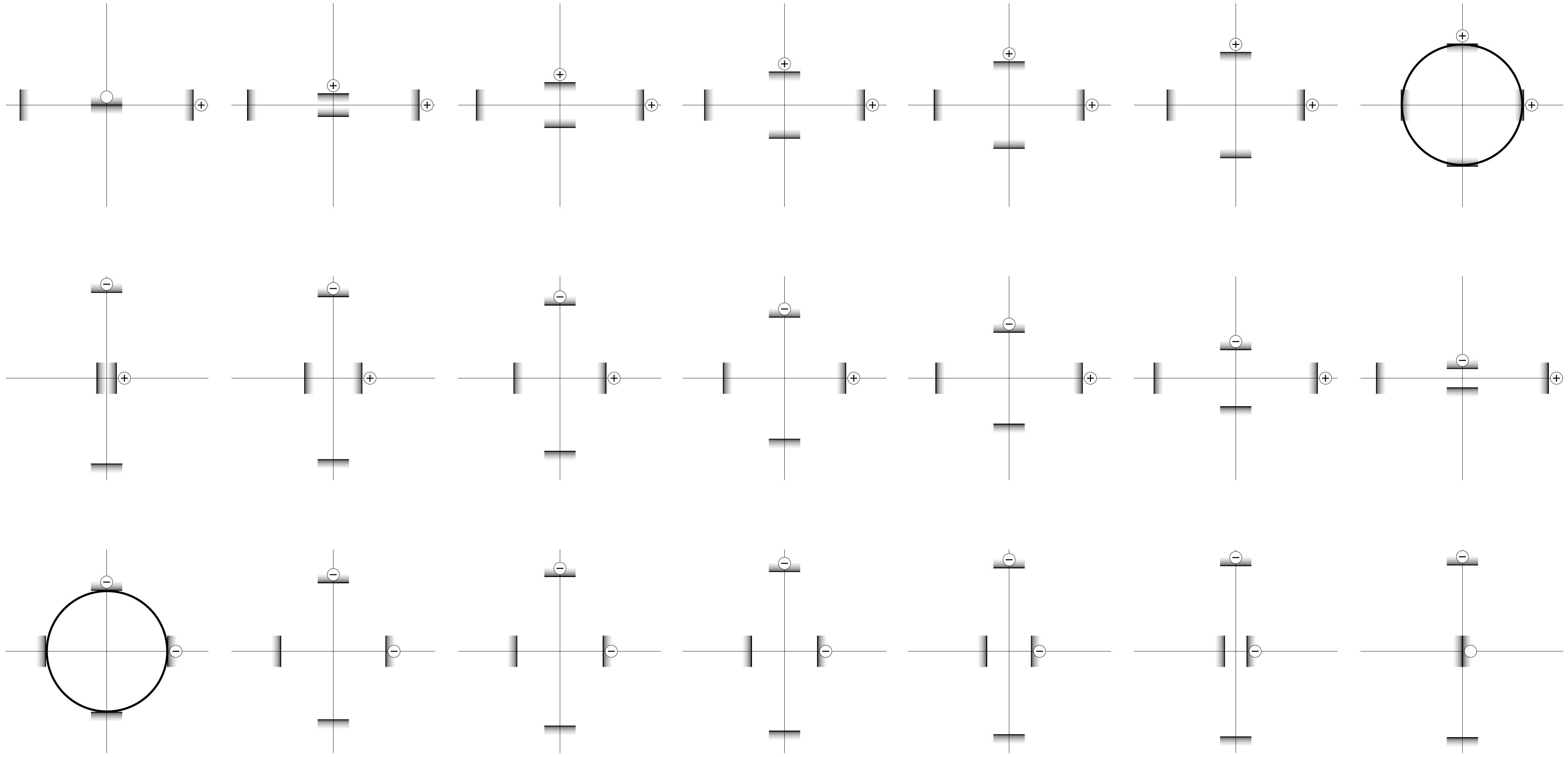

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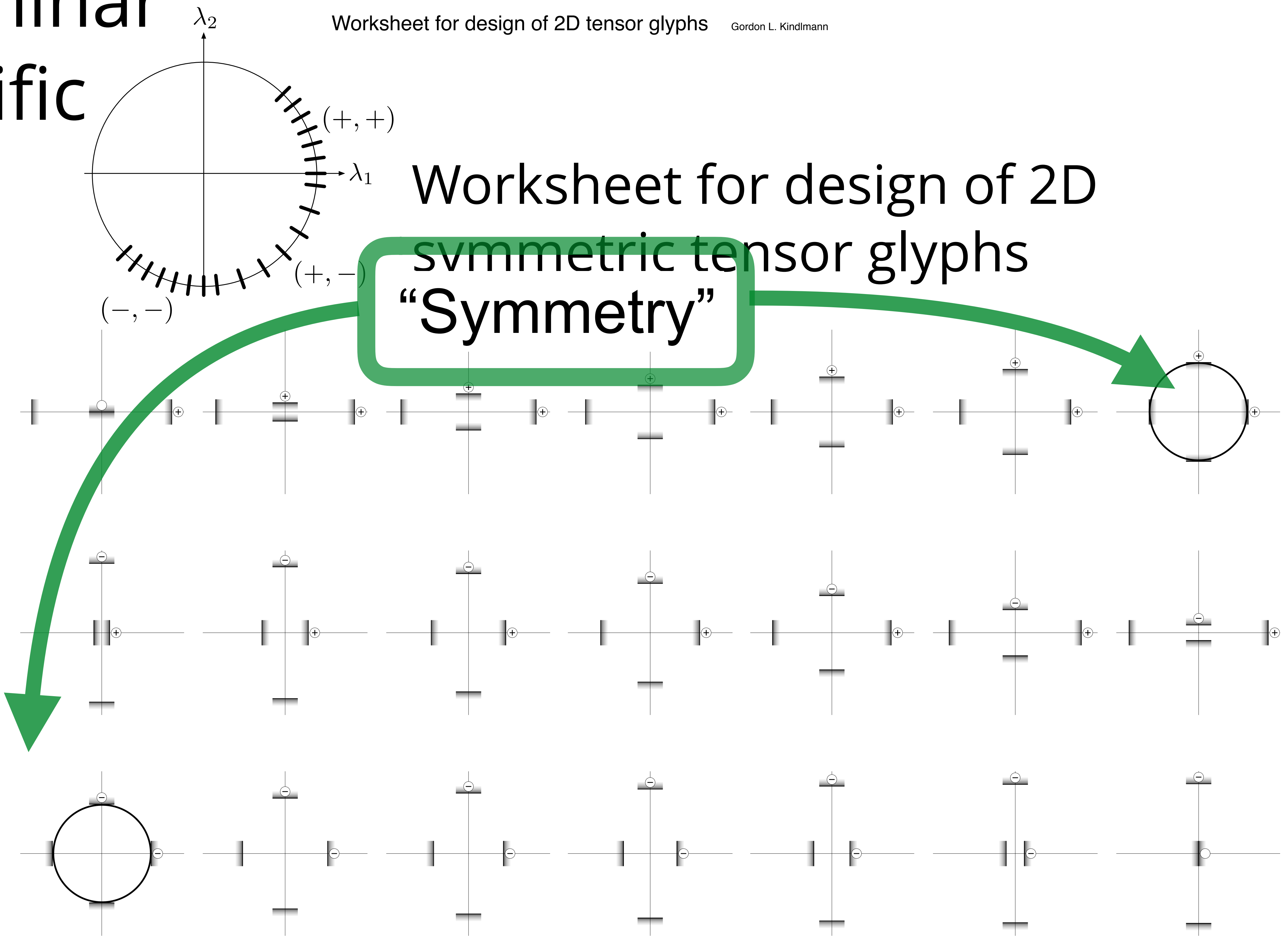

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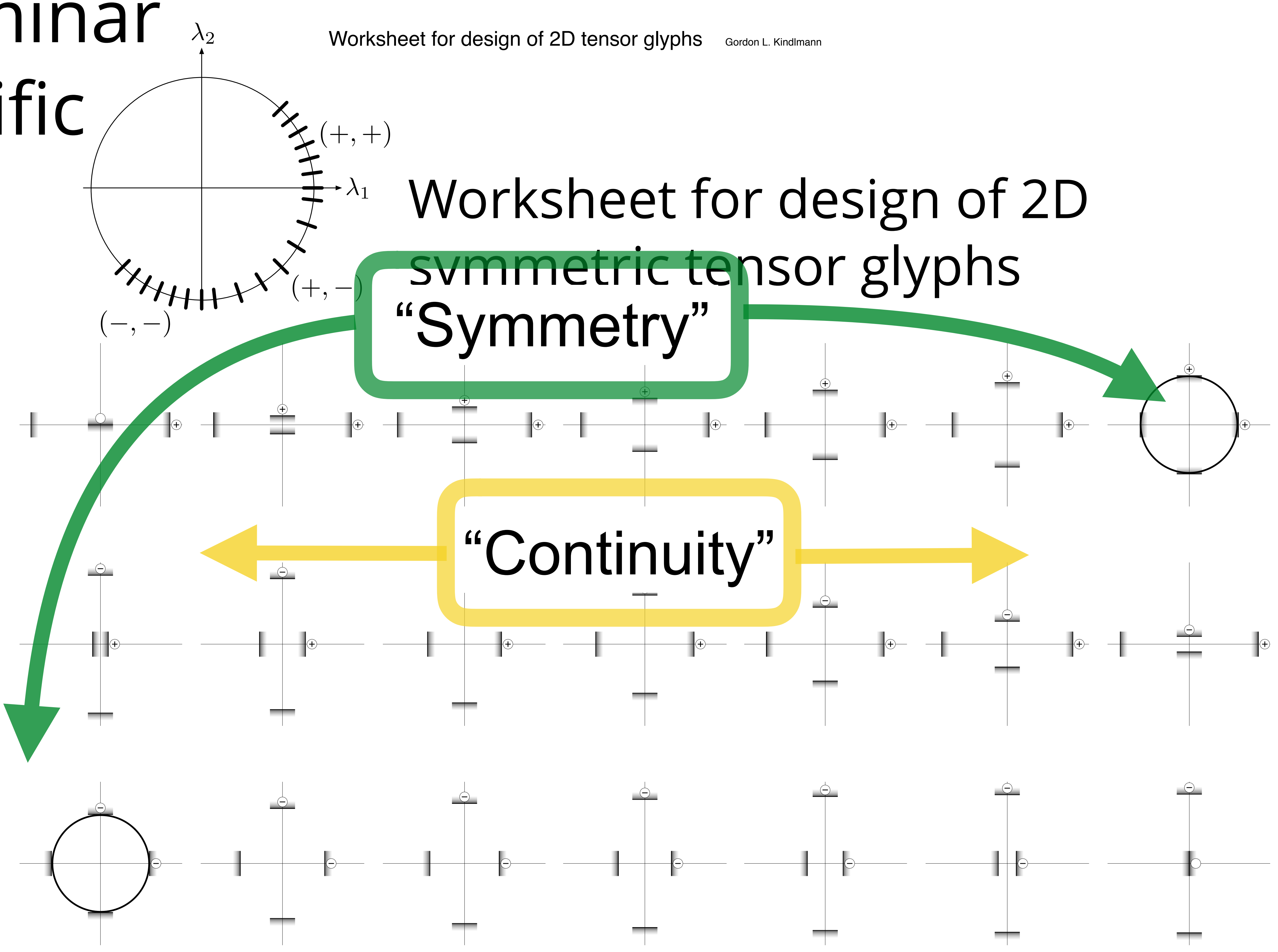

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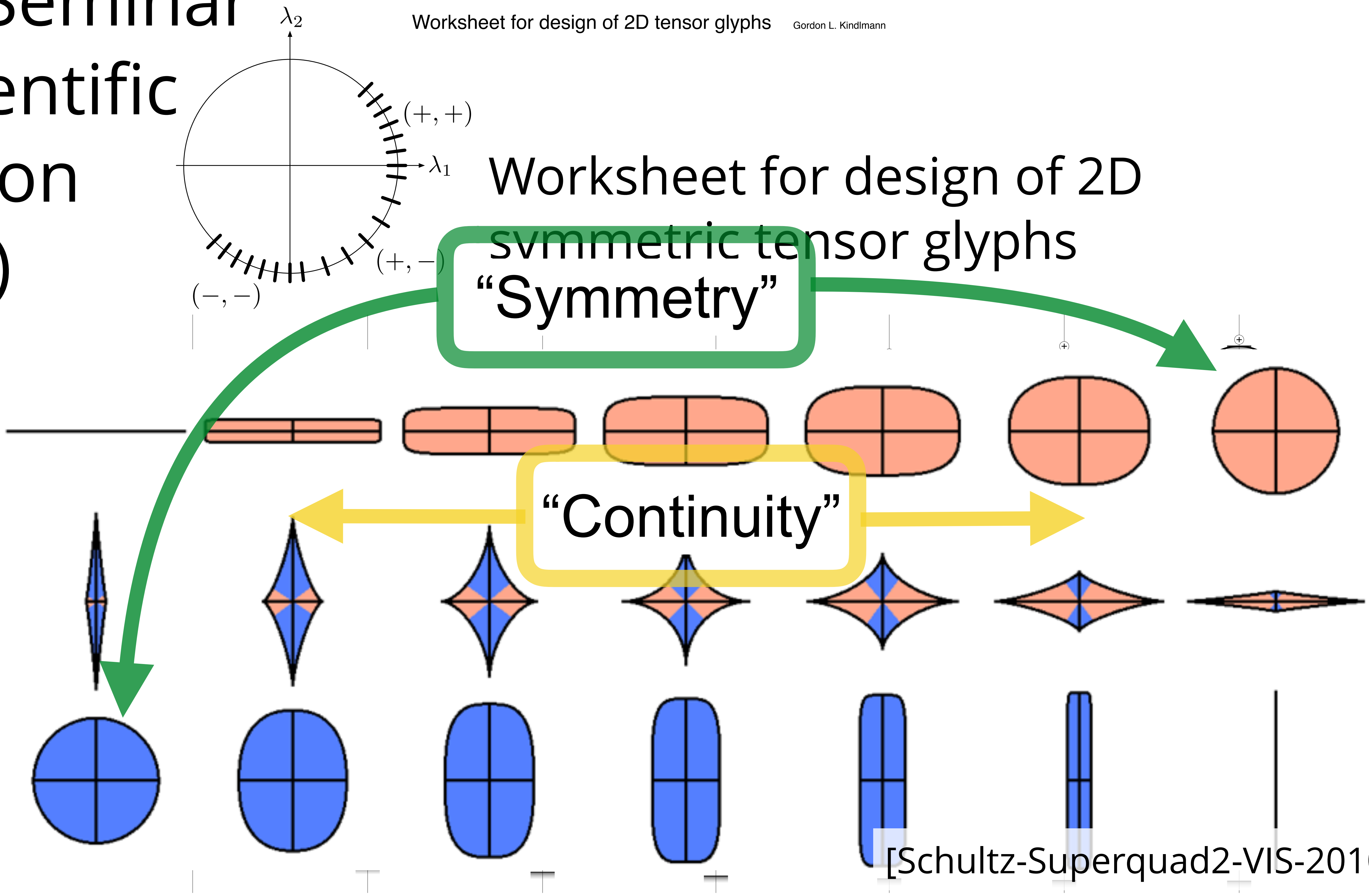

[Schultz-Superquad2-VIS-2010]

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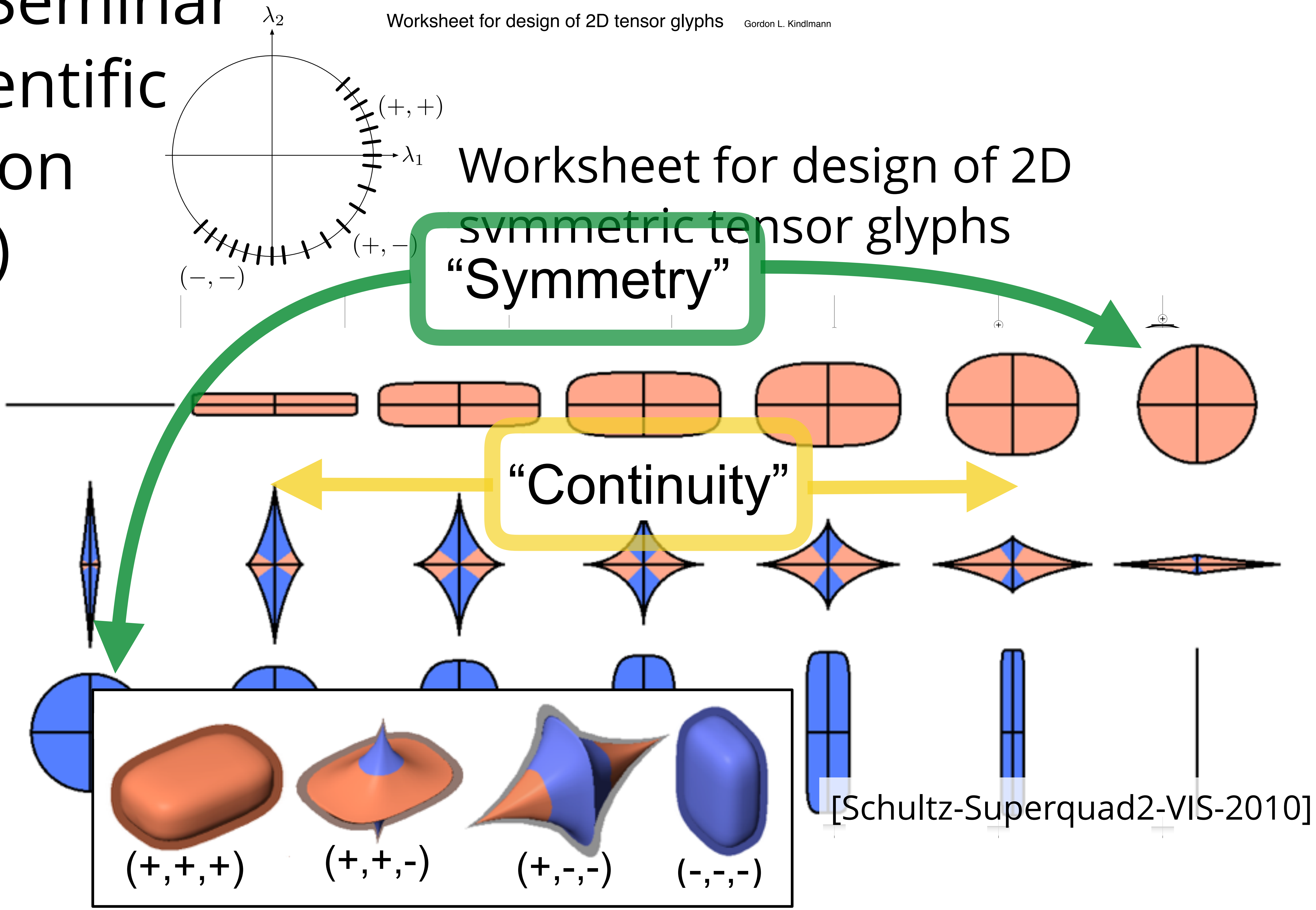

[Schultz-Superquad2-VIS-2010]

Mathematically abstracting vis design

T Transforms Data   **T'** Transform Vis

$$\begin{tikzcd} X \arrow[r, "V"'] \arrow[d, "T"'] & V(X) \arrow[d, "T'"'] \\ T(X) \arrow[r, "V"] & T'(V(X)) \end{tikzcd}$$

$$\approx$$

$$T \approx T' \Rightarrow V(T(X))$$

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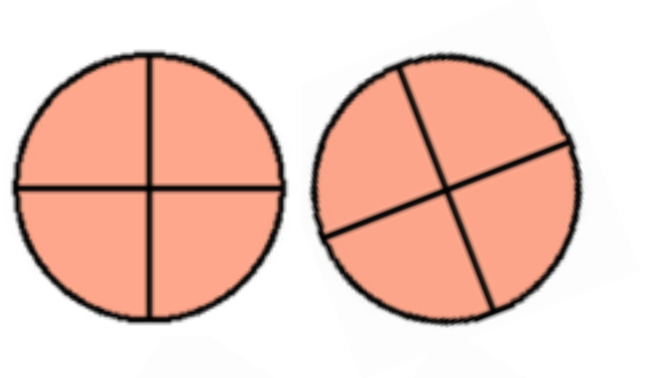
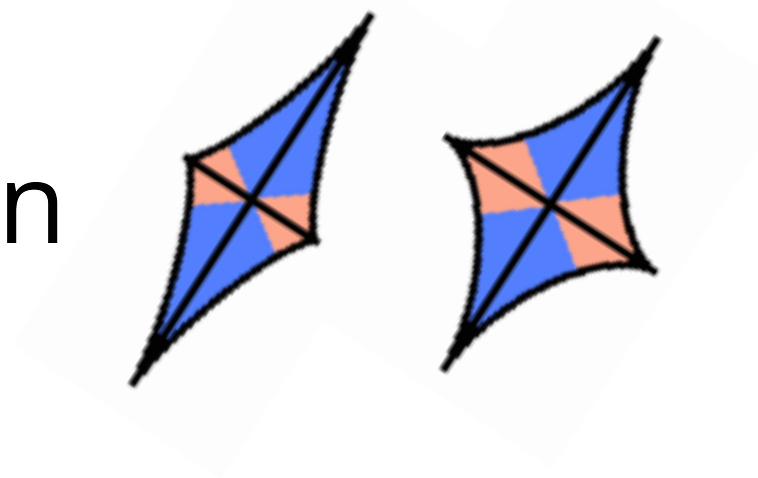
glyph property	T, T'	V(X), V(T(X))
"Symmetry"	Reflection, Rotation	
"Continuity"	Perturbation	

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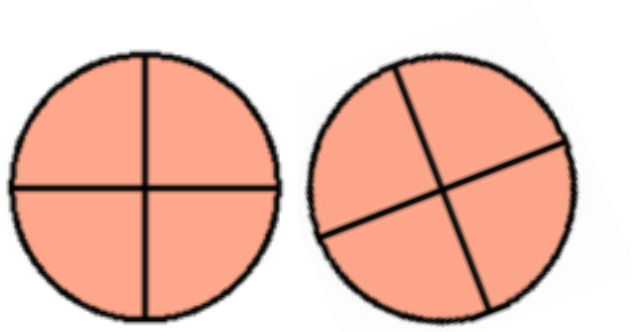
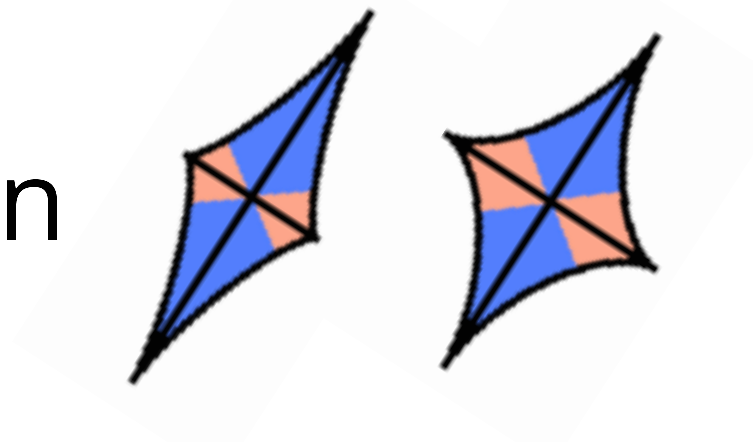
"Symmetry"	Reflection, Rotation	
"Continuity"	Perturbation	

[Kindlmann-AlgebraicVisDesign-VIS-2014]

Concrete Terminology for Problems:
Hallucinators, Confusers, Jumblers, Misleaders

Mathematically abstracting vis design

T Transforms Data   T’ Transform Vis

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X \arrow[r, "V"] \arrow[d, "T"'] & V(X) \arrow[d, "T'"] \\
T(X) \arrow[r, "V"] & T'(V(X))
\end{tikzcd}
$$

$$\approx$$

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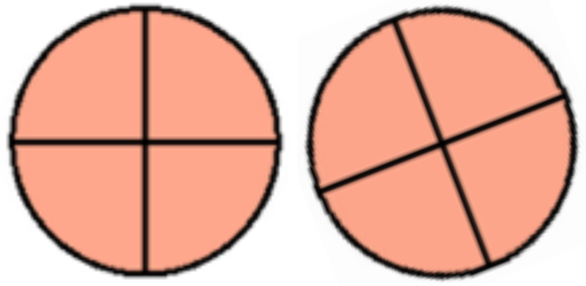
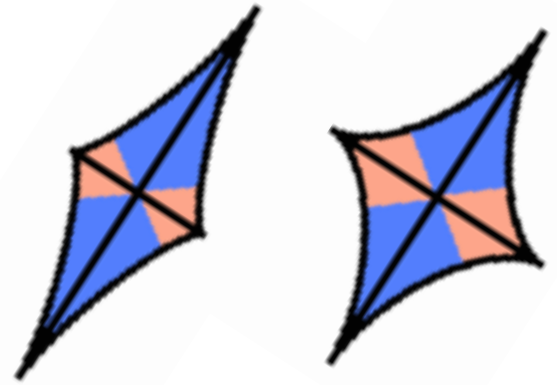
“Symmetry”	Reflection, Rotation	
“Continuity”	Perturbation	

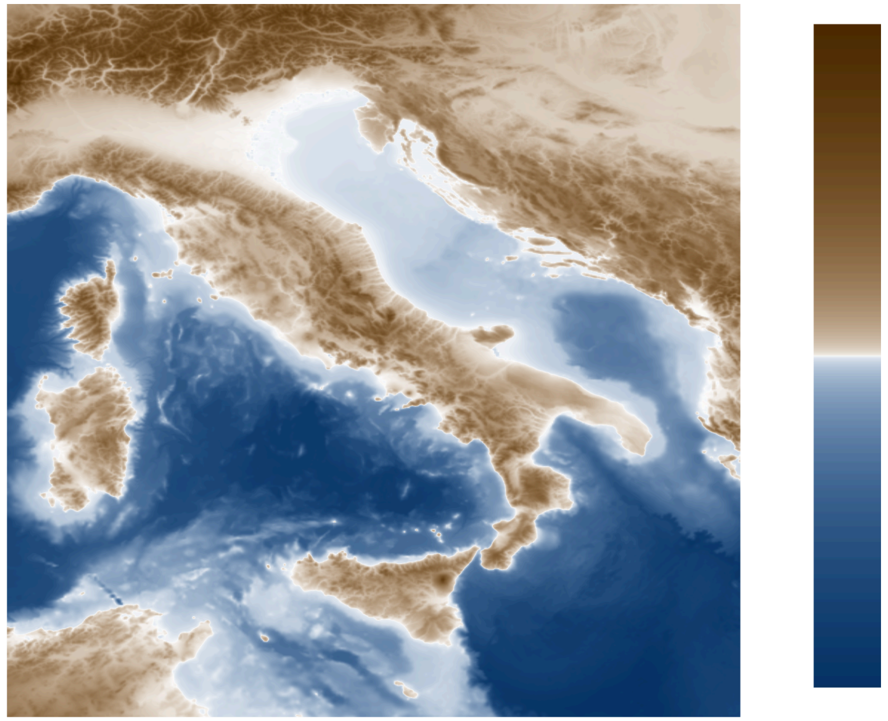
[Kindlmann-AlgebraicVisDesign-VIS-2014]

Concrete Terminology for Problems:
Hallucinators, Confusers, Jumblers, Misleaders
complements [Chen-InfoTheory-TVCG-2010]

Correspondence example: elevation colormap

Data: signed elevation relative to sea level

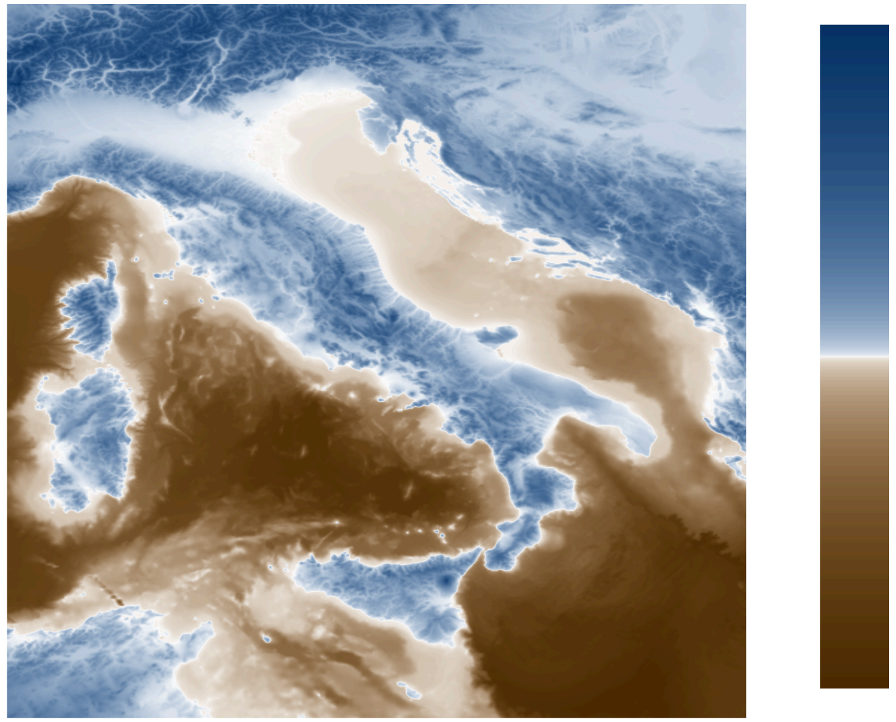
$$D \xrightarrow{v}$$



diverging colormap

α(e) = -e

$$D \xrightarrow{v}$$



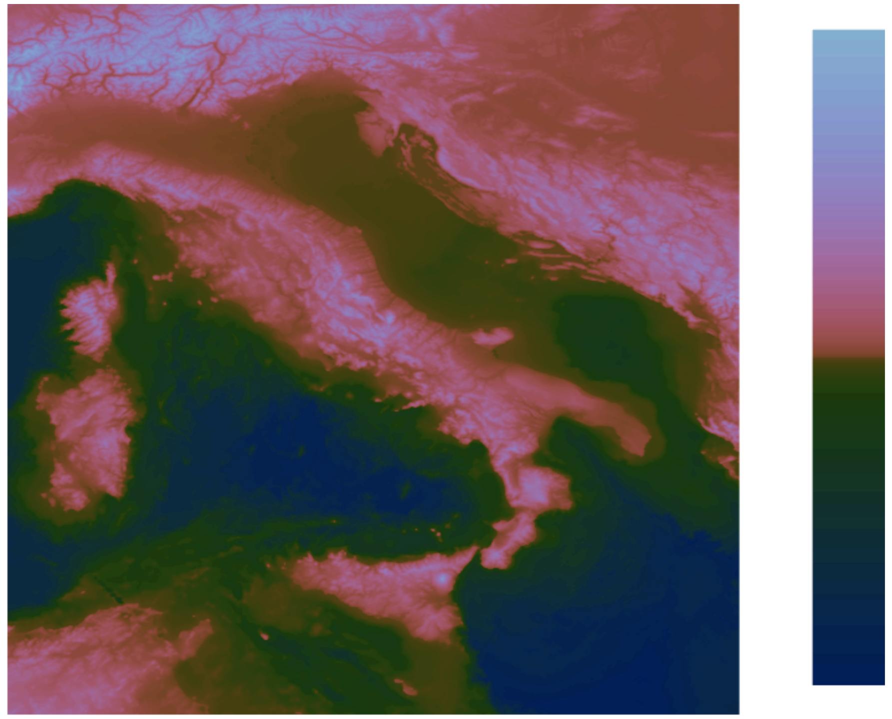
ω: negate hue

–v(e) ≈ v(–e)
colormapping commutes with negation

Correspondence example: elevation colormap

Data: signed elevation relative to sea level

$D \xrightarrow{v}$



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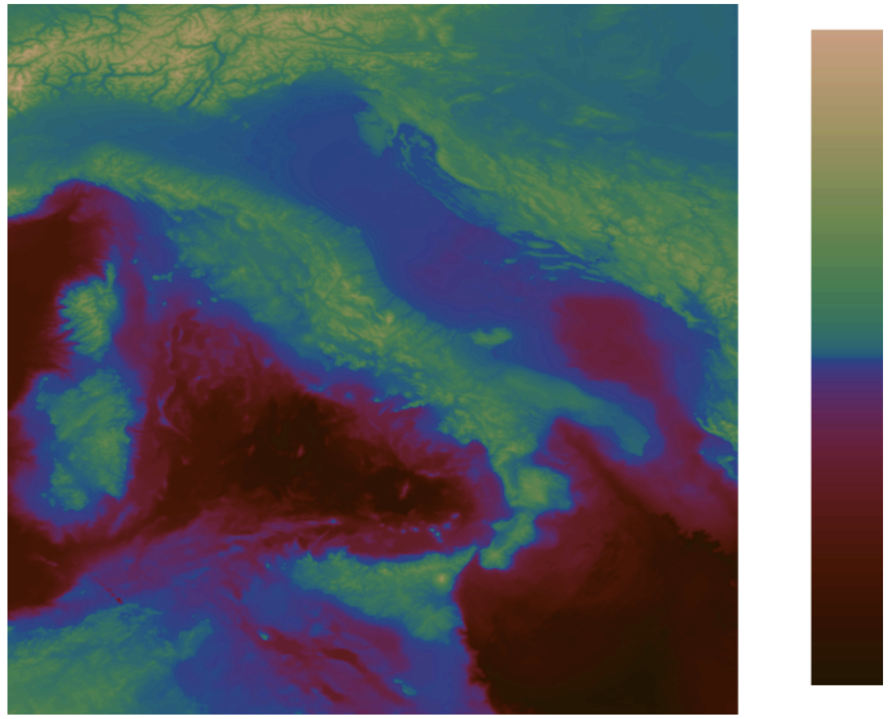

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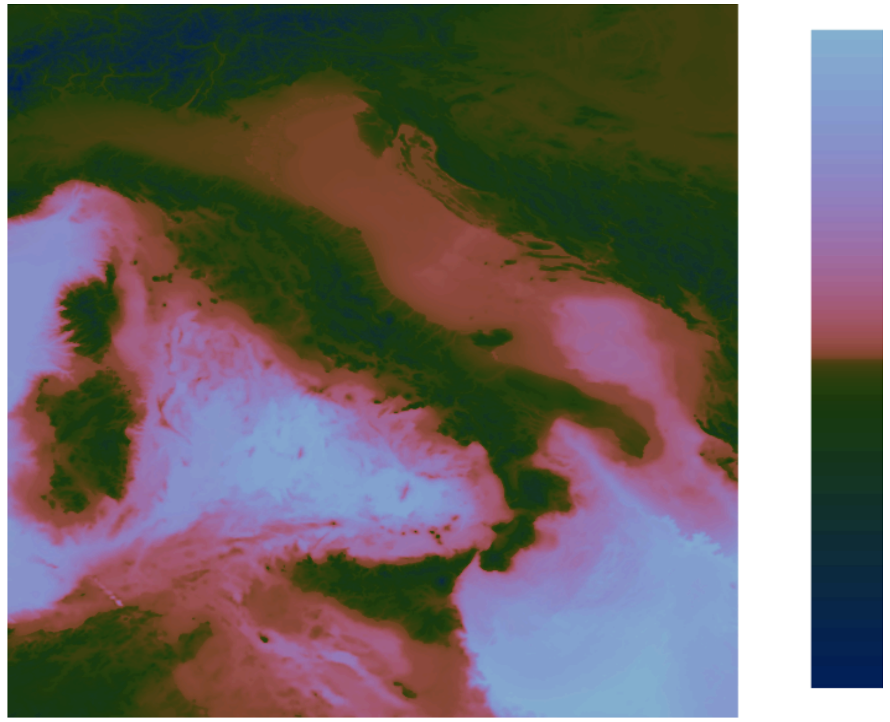

??

meaningful α not matched with perception: **“jumbler”**

(from seminar description)

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Mathematical foundations of visual data analysis. There is a rich tradition of ***mathematical and computational methods used in visualization***, such as topological approaches, feature extraction, numerical sampling and reconstruction methods, numerical integration, differential operators, filtering, dimension reduction, and applications of information theory, partly incorporating uncertainty. While all these methods have a solid mathematical foundation, a careful look at the relation between theories and their role in visual data analysis is needed.

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Anything else missing? Better way to organize?

Conclusions

Short of unifying "theory of vis", **need** accounting of <u>math **in** vis</u> (please help)

Why: principles empower vis students (more than a craft taught by apprenticeship)

<u>Math **of** vis</u> essential for Theory of vis

$\Rightarrow$ Either way, Vis needs Math

References

[Chen-InfoTheory-TVCG-2010] *An information-theoretic framework for visualization*. M Chen and H Jänicke. IEEE Transactions on Visualization and Computer Graphics, 16(6):1206–1215, 2010.

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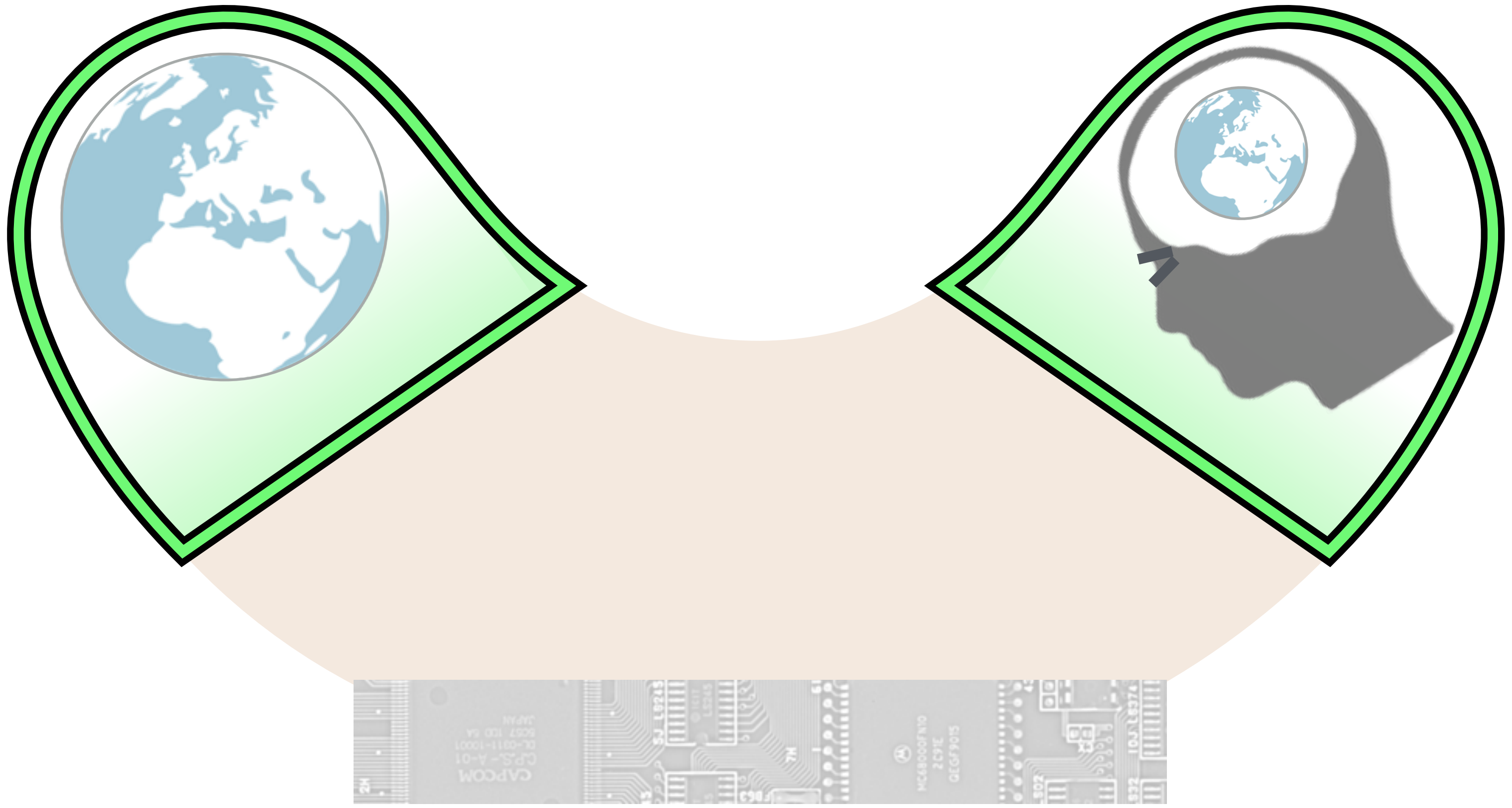

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