Quasi-Distances and Weighted Finite Automata

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Are neighbourhoods of a regular language also regular?
What is the state complexity of the neighbourhood of a regular language?

We use weighted finite automata to help us show the state complexity of the neighbourhood of a regular language (Salomaa, Schofield 2007).

1. Can additive additive WFAs recognize neighbourhoods with respect to additive quasi-distances?

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2. Is there a lower bound example for the state complexity of additive WFA languages over an alphabet with a constant number of symbols?

A distance is a function $d : \Sigma^* \times \Sigma^* \to [0, \infty)$ such that

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, w) + d(w, y)$

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If condition (1) is relaxed to $d(x, y) = 0$ if $x = y$, then d is a quasi-distance.

Islington $\rightarrow$ Eglin_ton

Montréal → Montreal

The neighbourhood of a language $L \subseteq \Sigma^*$ of radius $r \geq 0$ with respect to a distance measure d is the set of all words u with $d(w, u) \leq r$ for some $w \in L$,

$$E(L, d, r) = \{u \in \Sigma^* \mid (\exists w \in L)\, d(w, u) \leq r\}.$$

For which distances are neighbourhoods of regular languages regular for all radii $r \geq 0$?

A distance d on Σ^* is additive if for all factorizations $w = w_1 w_2$, we have for all $r \geq 0$

$$E(\{w\}, d, r) = \bigcup_{r_1 + r_2 = r} E(\{w_1\}, d, r_1) \cdot E(\{w_2\}, d, r_2)$$

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Theorem (Calude, Salomaa, Yu 2002)

Let d be an additive quasi-distance on Σ^* and $L \subseteq \Sigma^*$ be a regular language. Then $E(L, d, r)$ is regular for all $r \geq 0$.

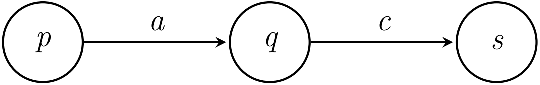
An additive weighted finite automaton is a 6-tuple $A = (Q, \Sigma, \gamma, \omega, q_0, F)$, where

- Q is the set of states
- Σ is the alphabet
- γ is the transition function
- ω is the weight function
- q_0 is the initial state
- F is the set of final states

Theorem (Salomaa, Schofield 2007)

Let A be an NFA, d an additive distance, and $r_0 \geq 0$. We can construct an additive WFA which recognizes the neighbourhood $E(L(A), d, r)$ for any $0 \leq r \leq r_0$.

This involves adding transitions with the appropriate weight.



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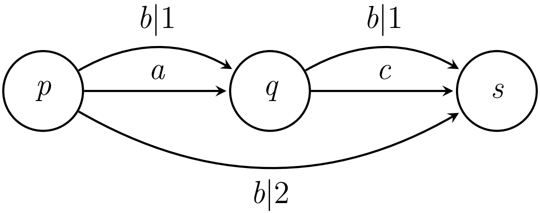

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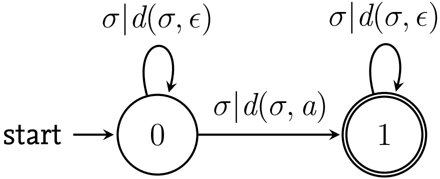
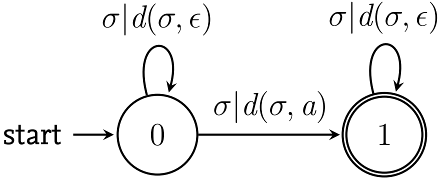
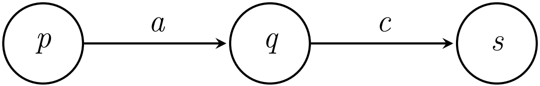
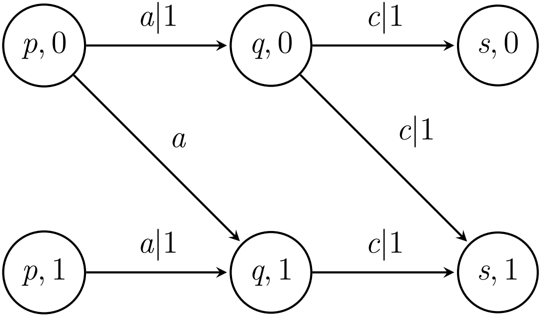
How do we construct an additive WFA for neighbourhoods with respect to quasi-distances?



$\sigma|d(\sigma,\epsilon)$
$\sigma|d(\sigma,\epsilon)$
$\sigma|d(\sigma,a)$
start
0
1


σ|d(σ, ϵ)
σ|d(σ, ϵ)
σ|d(σ, a)
start
0
1


p
a
q
c
s



Compute all-pairs shortest paths and consider the paths with weight at most r.

Theorem

Suppose that L has an NFA with n states and d is a quasi-distance. The neighbourhood of L of radius r can be recognized by an additive WFA having n states within weight bound r.

We can construct an equivalent DFA that recognizes a WFA with weight up to r. This requires at most $(r+2)^n$ states.

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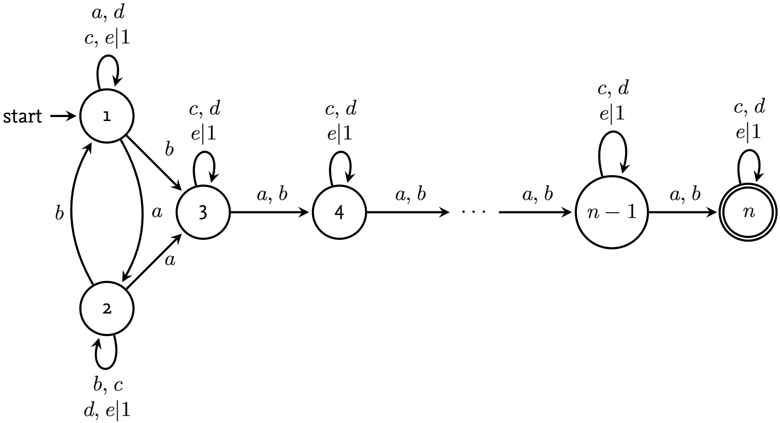
Theorem (Salomaa, Schofield 2007)

Let A be an additive WFA with integer weights and $r \in \mathbb{N}$. The language $L(A, r)$ can be recognized by a DFA having $(r+2)^n$ states.

We only need to keep track of the minimal weight computation that reaches each state.

$$(i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8)$$

$$(1, 1, 0, r+1, 4, r+1, r+1, r+1)$$



$$ac^{k_n}\, bd^{k_{n-1}}\, ac^{k_{n-2}} \cdots ac^{k_3}\, bd^{k_2}\, c^{k_1}, \text{ if } n \text{ is odd;}$$

$$abd^{k_n}\, ac^{k_{n-1}}\, bd^{k_{n-2}} \cdots ac^{k_3}\, bd^{k_2}\, c^{k_1}, \text{ if } n \text{ is even.}$$

Theorem

For $n, r \in \mathbb{N}$, there exist an n-state WFA with integer weights defined over a five-letter alphabet such that the state complexity of $L(A, r)$ is $(r + 2)^n$.

What is the state complexity of additive neighbourhoods?

- The WFA model implies an upper bound of $(r+2)^n$.
- Is there a matching lower bound?
- What is the state complexity when we consider specific distances?