State Complexity of Simple Splicing

Lila Kari and Timothy Ng
School of Computer Science, University of Waterloo
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Splicing systems

T. Head (1987)

A **splicing system** is a 3-tuple $H = (\Sigma, R, L)$ where

- Σ is a finite alphabet
- R is a set of rules
- L is the initial language

A **splicing rule** is a 4-tuple $r = (u_1,\ u_2;\ u_3,\ u_4)$ such that if $x = x_1 u_1 u_2 x_2$ and $y = y_1 u_3 u_4 y_2$, then

$$x \vdash_r y = x_1 u_1 u_4 y_2$$

The **language** of a splicing system $H = (\Sigma, R, L)$ is $R^*(L)$ where

$$R(L) = \{w \in \Sigma \mid (\exists\ x, y \in L,\ r \in R) \text{ such that } x \vdash_r y\}$$

and

- $R^0(L) = L$
- $R^i(L) = R^{\mathrm{i}}(L) \cup R(R^{i-1}(L))$
- $R^*(L) = \bigcup_{i \geq 0} R^i(L)$

Complexity of splicing systems

	Finite Ruleset	Regular Ruleset
Finite Initial Language	**Locally testable** (T. Head, 1987)	**Recursively enumerable** (T. Head, Gh. Păun, D. Pixton, 1997)
Regular Initial Language	**Regular** (K. Culik II and T. Harju, 1991)	**Recurisvely enumerable** (Gh. Păun, 1996)

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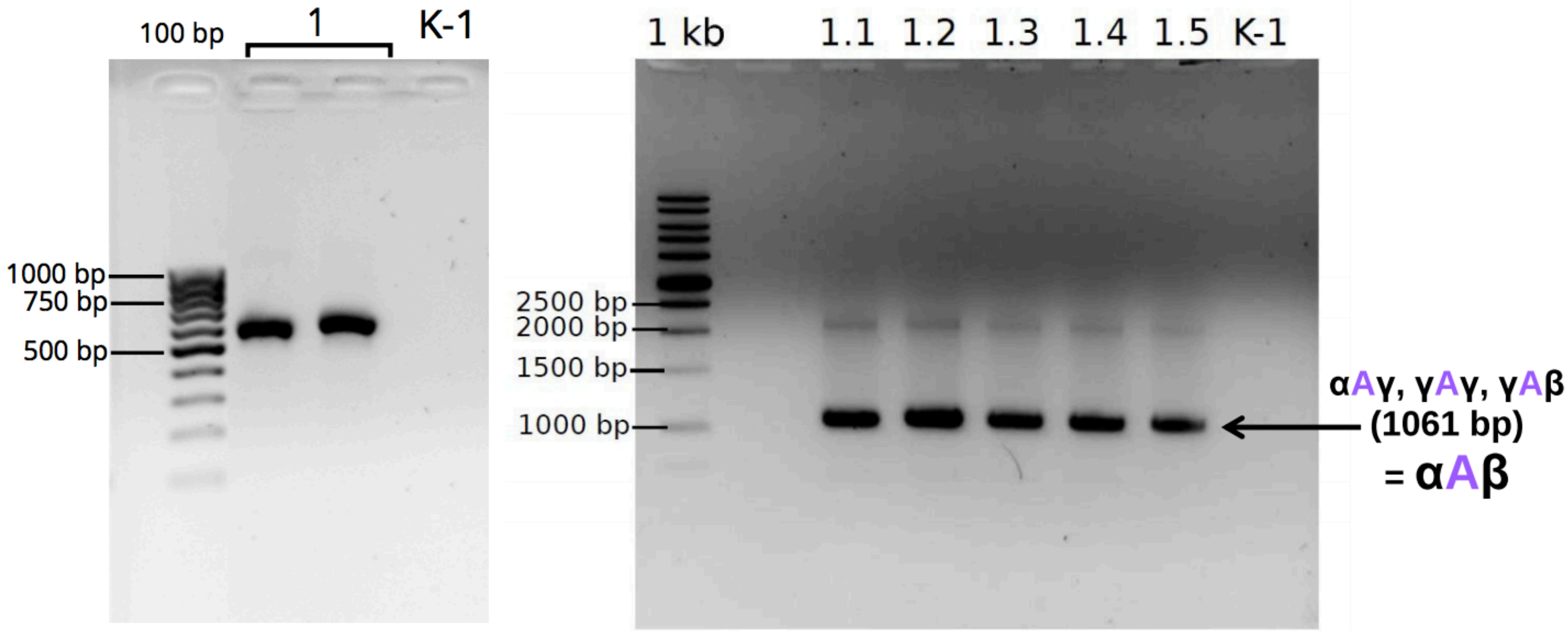
Simple splicing

A. Mateescu, Gh. Păun, G. Rozenberg, A. Salomaa (1998)

A splicing rule r is **simple** if $r = (u_1, ε; u_3, ε)$ where $u_1 = u_3$ and $|u_1| = 1$.

A splicing system with only simple rules is a **simple splicing system**.

A simple splicing system is denoted by $H = (\Sigma, M, L)$ where $M \subseteq \Sigma$. Then $a \in M$ means $(a, ε; a, ε)$ is a rule in H.



Franco, Bellamoli, Lampis (2017)

Word Blending

S.K. Enaganti, L. Kari, T. N., Z. Wang (2018)

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$$
\begin{aligned}
u \bowtie v = \{\alpha w \beta \mid & u = \alpha w \gamma_1, v = \gamma_2 w \beta, \\
& \alpha, \beta, w, \gamma_1, \gamma_2 \in \Sigma^*\}
\end{aligned}
$$

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$$
\begin{aligned}
u \bowtie v = \{\alpha a \beta \mid & u = \alpha a \gamma_1, v = \gamma_2 a \beta, \\
& \alpha, \beta, \gamma_1, \gamma_2 \in \Sigma^*, a \in \Sigma\}
\end{aligned}
$$

Descriptional complexity measures for splicing systems

- Radius (Gh. Păun, 1996)
- Size of initial language (A. Mateescu et al., 1998)
- Size of grammar (A. Mateescu et al., 1998)
- Number/length of rules (R. Loos et al., 2007)
- Size of nondeterministic finite automaton (R. Loos et al., 2007)

The **state complexity of a regular language** is the number of states in its minimal deterministic finite automaton.

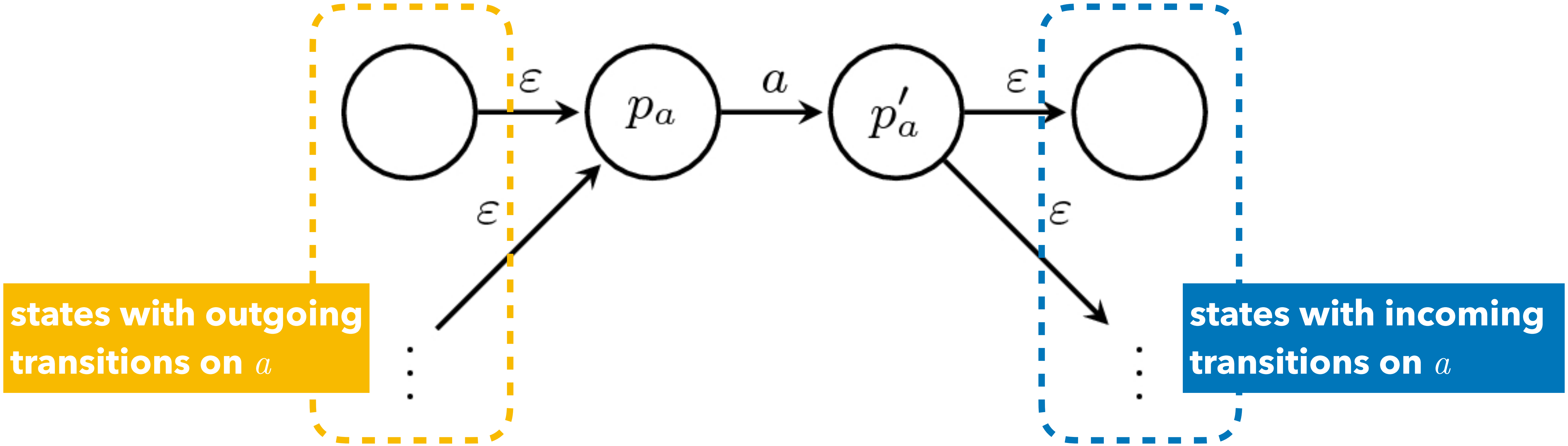
The **state complexity of an operation** is the worst-case state complexity of the language resulting from the operation, as a function of the state complexity of the operands.

For a **simple** splicing system with initial language L with state complexity n

$$\begin{cases} 2^n - 1 & \text{if } L \text{ is regular,} \\ 2^{n-2} + 1 & \text{if } L \text{ is finite,} \end{cases}$$

Let $H = (\Sigma, M, L)$ be a simple splicing system and let A **be the DFA for** L. From A, we will **construct an NFA** that recognizes the language of H.

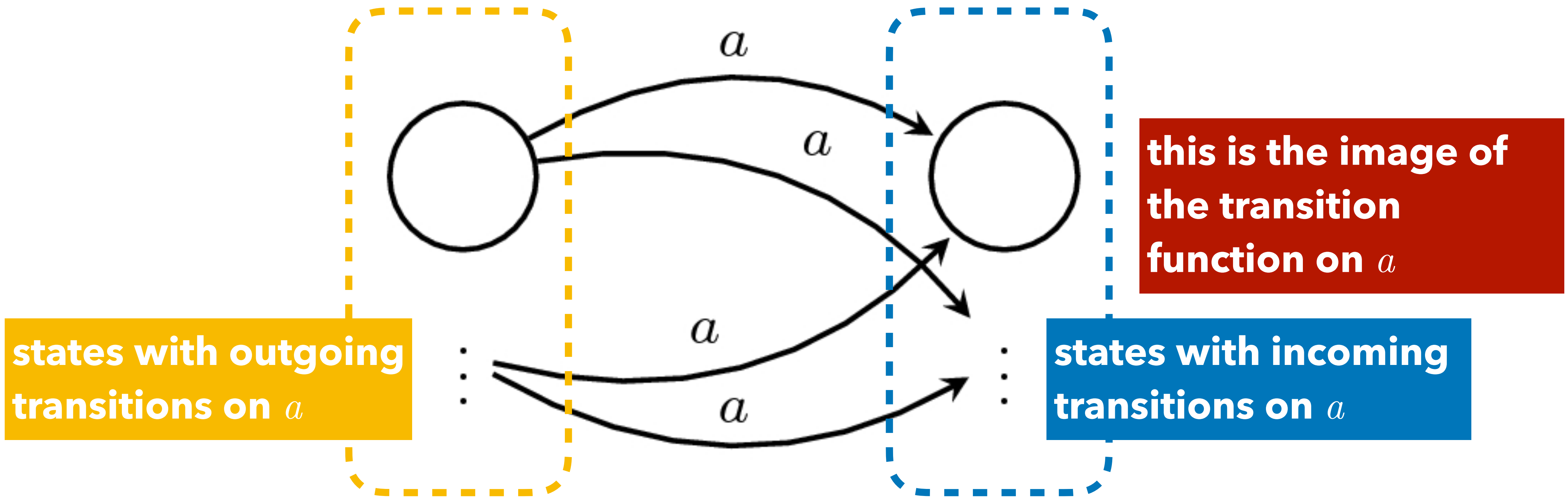
For each symbol $a \in M$, construct a **bridge**:



Add ε-transitions:

- from each state in A with outgoing transitions on a to p_a, and
- from p'_a to all states of A with incoming transitions on a

Then perform **ε-transition removal**:

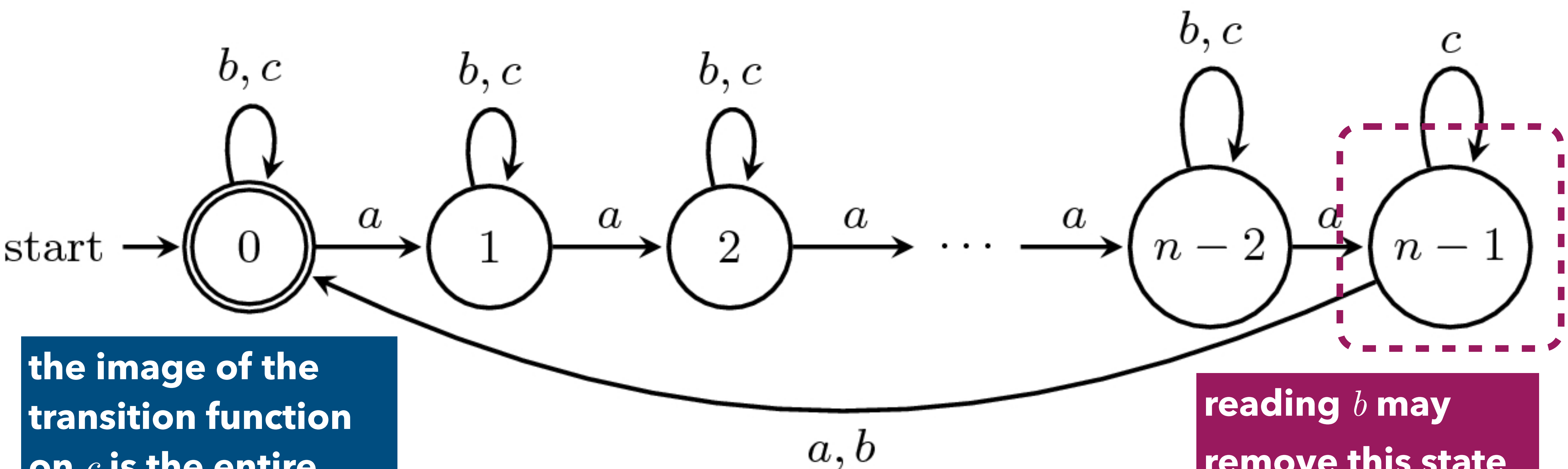


The new states p_a and p'_a disappear and collapse into transitions between states of A on a.

Since we begin with an n-state DFA, this process results in an **n-state NFA** after removing ε-transitions.

This gives an **upper bound of $2^n - 1$** reachable (non-empty subsets) states via the subset construction.

The upper bound is reachable via the simple splicing system $(\{a,b,c\},\{c\},L_n)$, where L_n is recognized by the following DFA



the image of the transition function on c is the entire state set

reading b may remove this state

The upper bound is lower with a finite initial language

- Consider a simple splicing system (Σ, M, I), where I **is a finite language with state complexity** n
- Since I is finite, its DFA A, is **acyclic**. Then the initial state q_0 of A has no incoming transitions so **the only reachable subset containing** q_0 **is** $\{q_0\}$.
- Since I is finite, A must contain a **sink state**.
- This gives a total of $2^{n-2} - 1 + 2$ **states**.

For a simple splicing system with initial **finite** language L with state complexity n **over** k **symbols**

$$\begin{cases} 2 & \text{if } k = 1, \\ 2n-3 & \text{if } k = 2, \\ 2^{\frac{n}{2}} + 1 & \text{if } k = 3 \text{ and } n \text{ is even}, \\ 3 \cdot 2^{\frac{n-3}{2}} + 1 & \text{if } k = 3 \text{ and } n \text{ is odd}, \\ 2^{n-2} + 1 & \text{if } k \geq 2^{n-3}. \end{cases}$$

Lemma. If $a \in M$, then $\mathrm{im}\ \delta_a{}'$ contains exactly the sink state and $\mathrm{im}\ \delta_a$.

In other words, if $a \in M$, then reading a will take the DFA to either exactly one subset or the sink state.

To reach the upper bound for finite initial languages

- Let q_1 be a state reachable **directly by the initial state** and no other state; this state must exist since the DFA is acyclic.

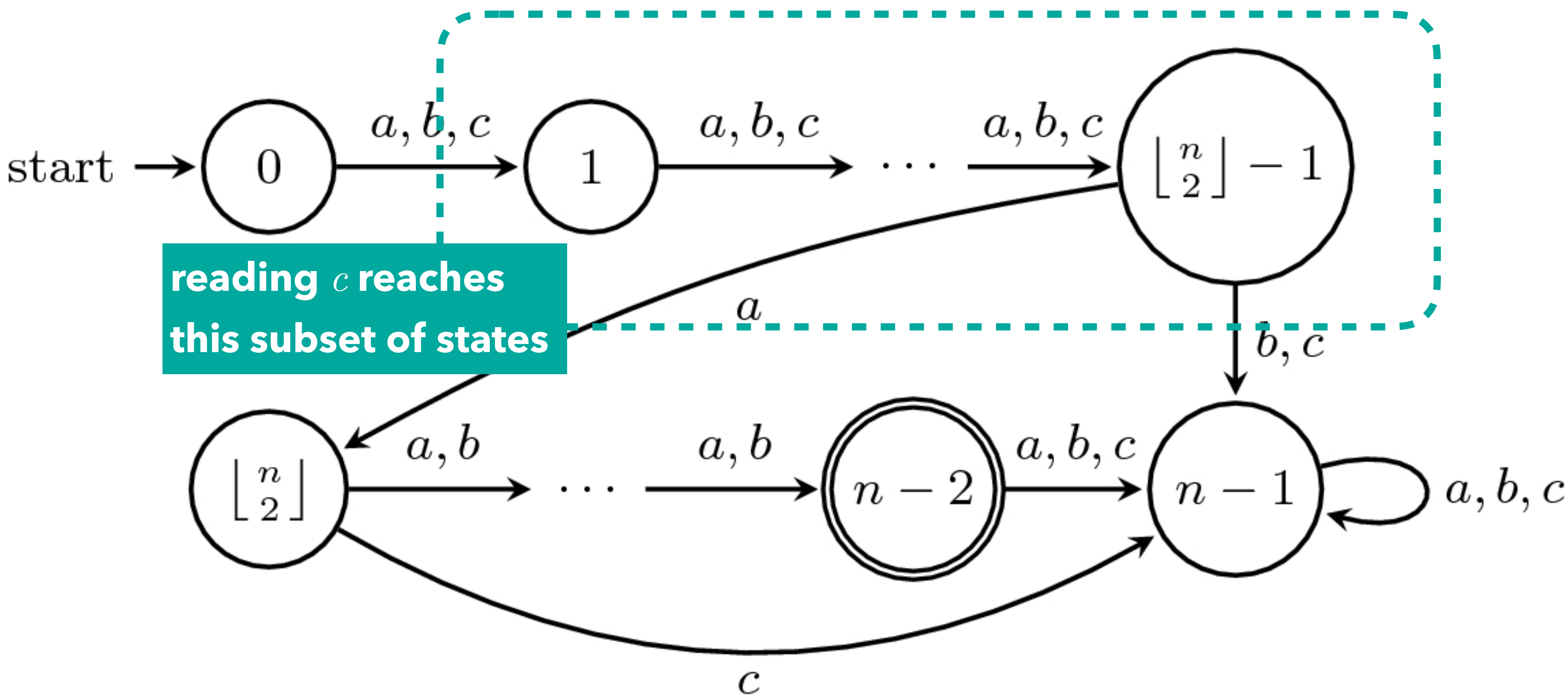
- If $q_1 \in S$, then S is reachable only if it is **the image of δ_a for some $a \in M$**.

- Since there are up to 2^{n-3} subsets that contain q_1, to reach each of these subsets, **there must be one $a \in M$ for each**.

If $k = 2$

- If a, b are not in M, then we just **have L**.
- If a, b are both in M, then we have **at most two** reachable subsets.
- If $a \in M$ and b is not, then to maximize the number of reachable states, we must have $\delta_b(i) = i+1$ and $|\mathrm{im}\ \delta_a| = 2$. This gives us **at most $2n-3$ states**.

For $k = 3$, the upper bound is reached by $(\{a,b,c\},\{c\},I_n)$ where I_n is recognized by the DFA below.



A splicing rule r is **semi-simple** if $r = (u_1, \varepsilon; u_3, \varepsilon)$ with $|u_1| = |u_3| = 1$.

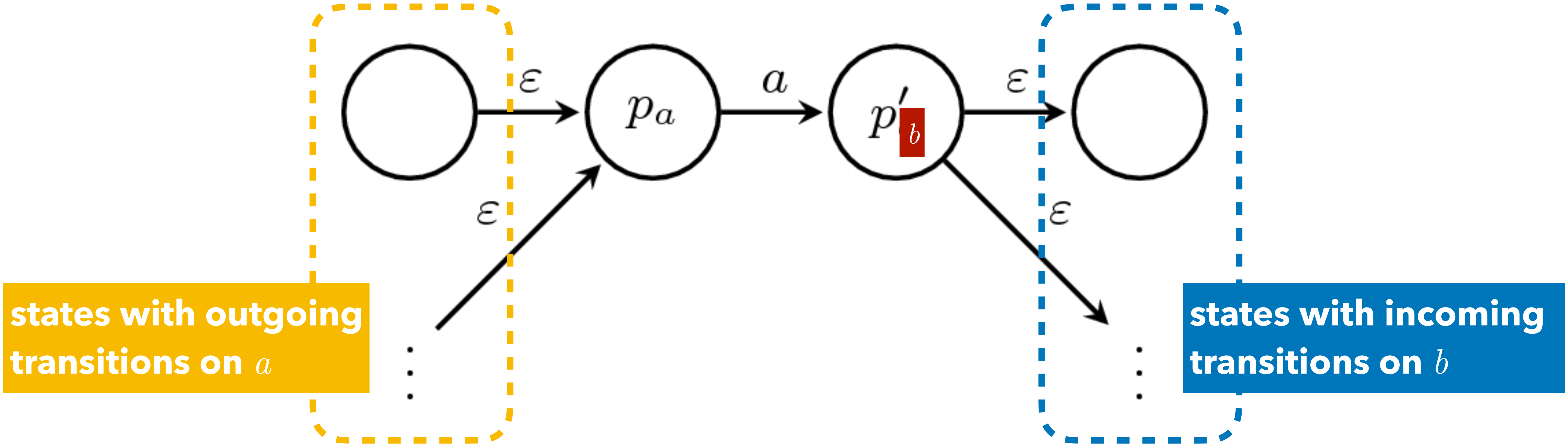
A splicing system with only simple rules is a **semi-simple splicing system**.

A semi-simple splicing system is denoted by $H = (\Sigma, M^{(2)}, L)$ where $M^{(2)} \subseteq \Sigma \times \Sigma$. Then $(a,b) \in M^{(2)}$ means $(a, \varepsilon; b, \varepsilon)$ is a rule in H.

Semi-simple splicing

E. Goode and D. Pixton (2001)

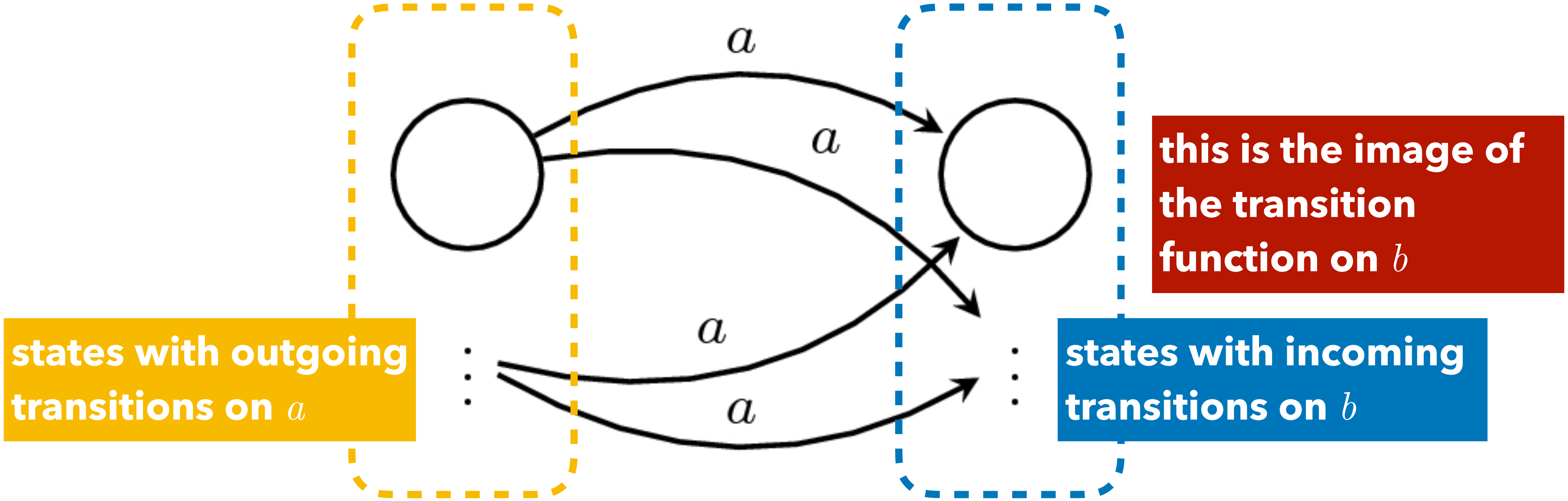
For each pair $(a, b) \in M^{(2)}$, construct a **bridge**:



Add ε-transitions:

- from each state in A with outgoing transitions on a to p_a, and
- from p'_b to all states of A with incoming transitions on b

Then perform **ε-transition removal**:

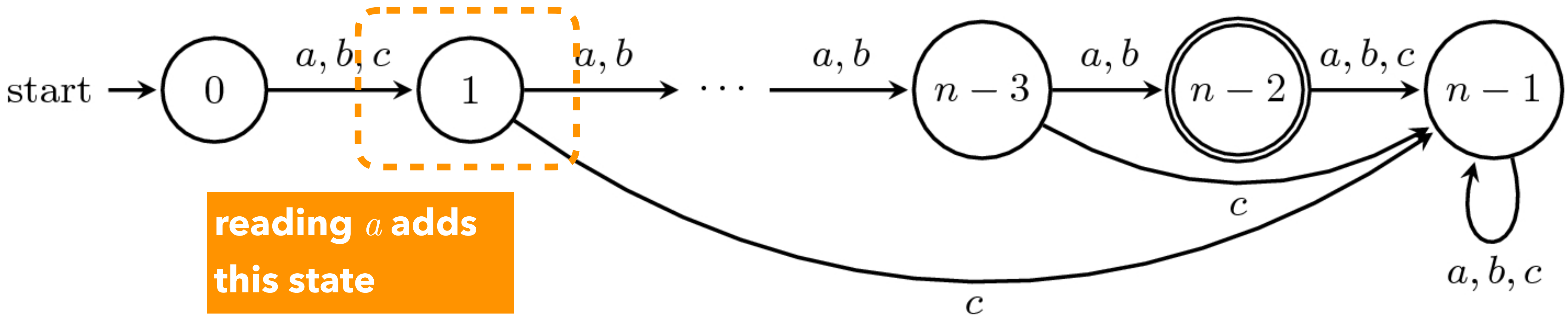


The new states p_a and p'_b disappear and collapse into transitions between states of A on a.

This construction shows that **semi-simple** splicing systems with regular and finite initial languages have the **same upper bound** for state complexity.

For semi-simple splicing systems with a regular initial language, this upper bound is reached by the **same lower bound witness** for simple splicing systems.

The upper bound for semi-simple splicing systems with a **finite initial language** can be reached via $(\{a,b,c\},\{(a,c)\},I_n)$, where I_n is recognized by the following DFA



For $M \subseteq \Sigma \times \Sigma$ we define the operation on two strings u,v by

$$u \diamond_M v = u'av'$$

if $u = u'a$ and $v = bv'$ for $(a,b) \in M$ and u',$v' \in \Sigma$; and is undefined otherwise.

The **crossover operation** can be defined in terms of this operation extended to languages by

$$L_1 \sharp_M L_2 = \mathrm{pref}(L_1) \diamond_M \mathrm{suff}(L_2)$$

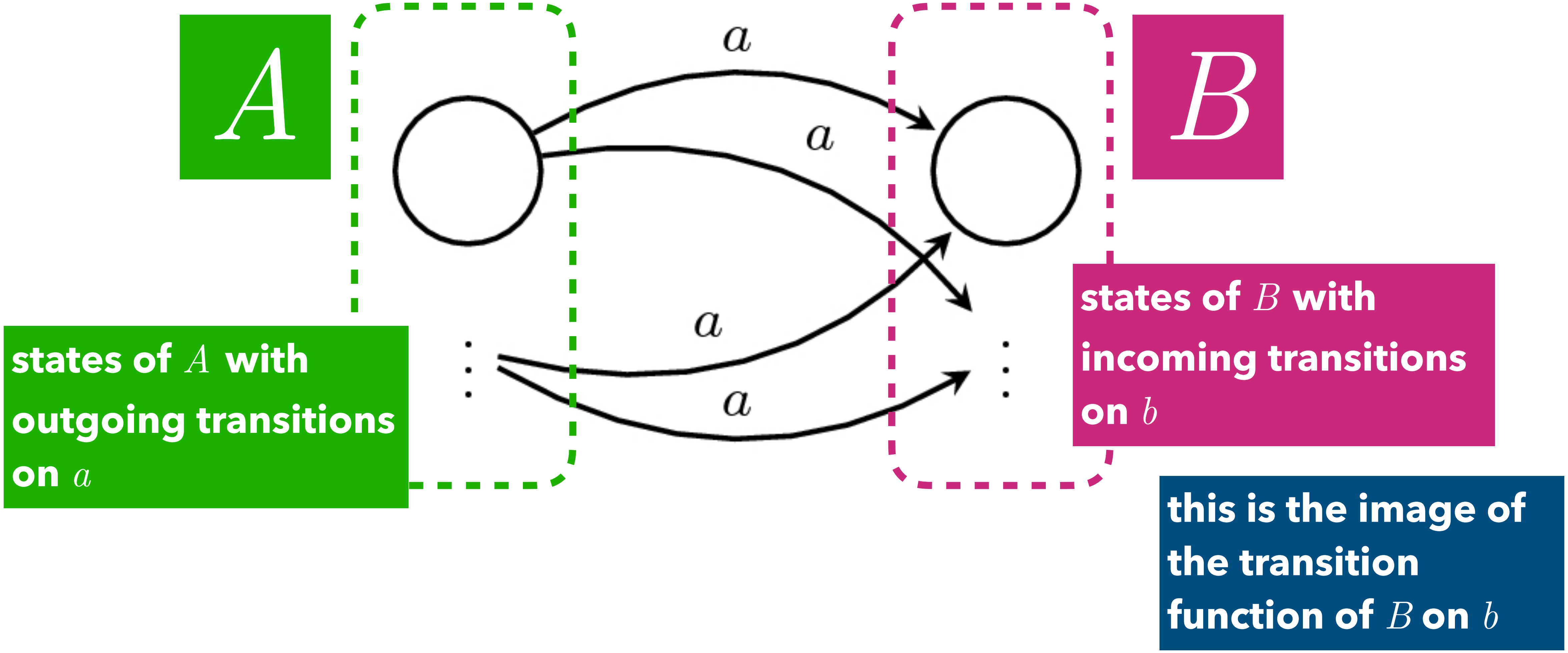
Crossover

A. Mateescu et al. (1998), R. Ceterchi (2006)

The crossover operation is used for the **algebraic characterization** of simple and semi-simple splicing.

The operation can be thought of as a **single step** of simple or semi-simple splicing.

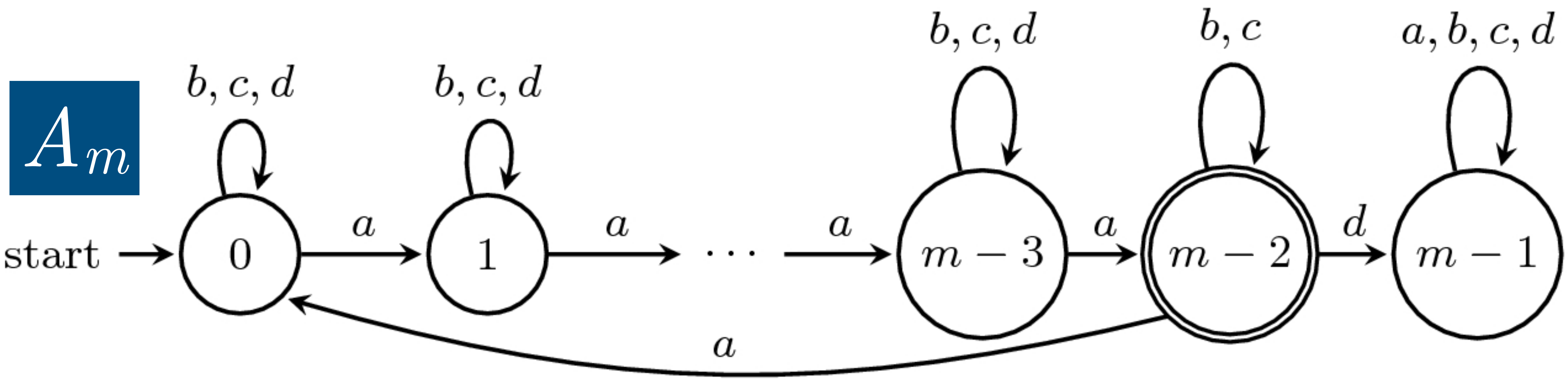
For two DFAs A and B, if $(a,b) \in M$, then for each state of A with outgoing transitions on a, add transitions on a to all states in B with incoming transitions on b.



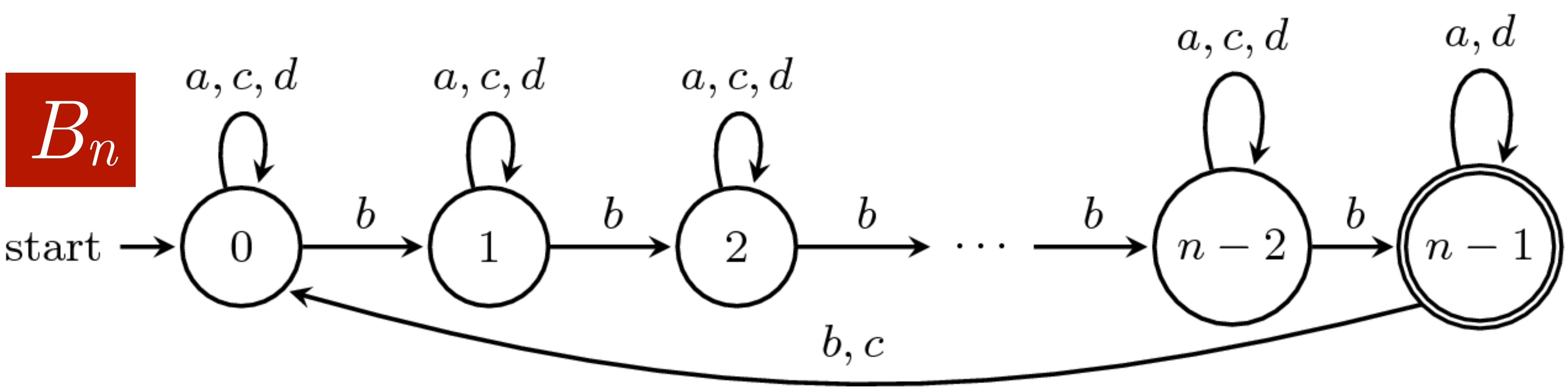
This gives **at most $m \times 2^n$ states**.

$$M = \{(d,d)\}$$

A_m


start
0
1
$m-3$
$m-2$
$m-1$
b, c, d
b, c
a, b, c, d
a
d
$\cdots$

B_n


start
0
1
2
$n-2$
$n-1$
a, c, d
a, d
b
b, c
$\cdots$

Conclusion

- State complexity for **simple** splicing systems with **regular** initial languages
- State complexity for **simple** splicing systems with **finite** initial languages defined over alphabets of size **1, 2, 3, and $\geq 2^{n-3}$**
- State complexity of **semi-simple** splicing systems with **regular** and **finite** initial languages
- State complexity of the **crossover** operation on regular languages

Open problems

- State complexity for **other simple and semi-simple splicing systems (2,4; 2,3; 1,4)** with finite and regular initial languages.

- State complexity of simple splicing systems with finite initial languages over **alphabets of size k for $3 < k < 2^{n-3}$**.

- State complexity of **k-limited splicing systems**, for $k = 1, 2, \ldots$.

Thank you