# State Complexity of Pseudocatenation

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Adenine

Cytosine

Guanine

Thymine

ACGTACGTAGCTGCATCGATCAG

ACGTACGTAGCTGCATCGATCAG
TGCATGCATCGACGTAGCTAGTC

ACGTACGTAGCTGCATCGATCAG

TGCATGCATCGACGTAGCTAGTC

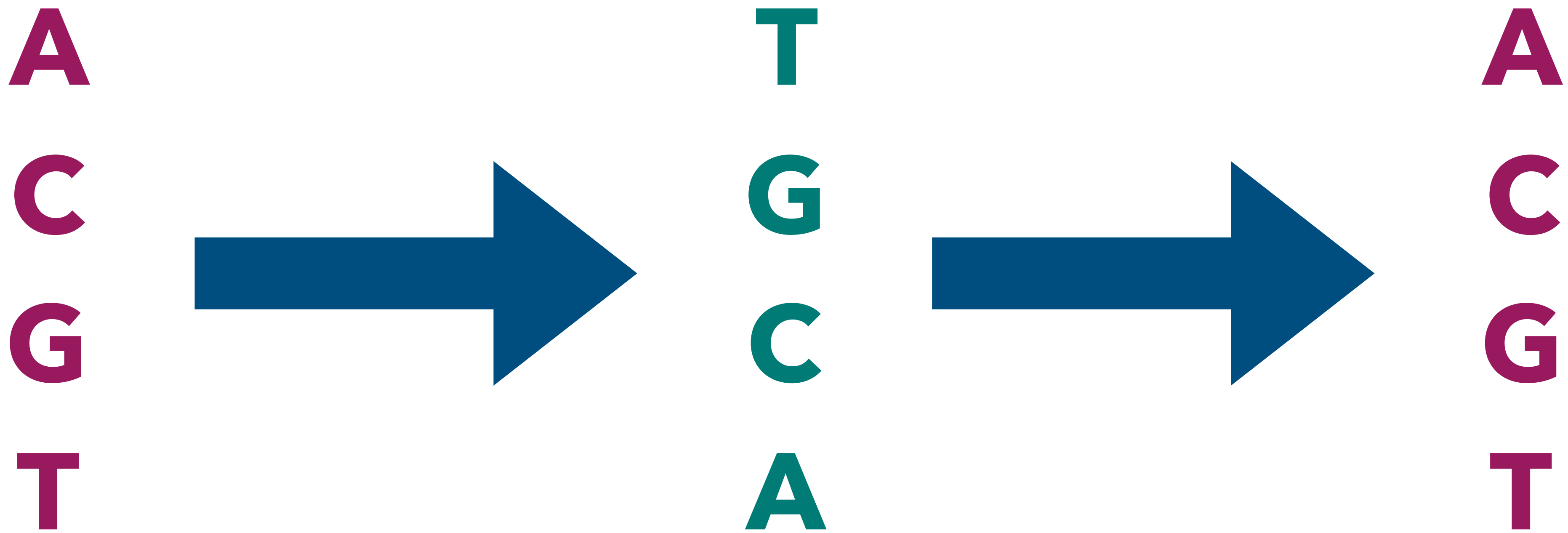

ACGTACGTAGCTGCATCGATCAG
TGCATGCATCGACGTAGCTAGTC

ACGTACGTAGCTGCATCGATCAG

TGCATGCATCGACGTAGCTAGTC

CTGATCGATGCAGCTACGTACGT


ACGTACGTAGCTGCATCGATCAG
TGCATGCATCGACGTAGCTAGTC
CTGATCGATGCAGCTACGTACGT




# Involution

ACGTACGTAGCTGCATCGATCAG

TGCATGCATCGACGTAGCTAGTC

$$\theta(a_1 a_2 \cdots a_{n-1} a_n) = \theta(a_n)\theta(a_{n-1}) \cdots \theta(a_2)\theta(a_1)$$

# Antimorphism


# Palindromicity

A. de Luca and A. De Luca (2006)

**AGGTACGTACCTGTACCTAGGTACAGGTAC**

$$w = x_1 x_2 \cdots x_k, x_i \in \{x_1, \theta(x_1)\}$$

$$w \in u\{u, \theta(u)\}^*$$

# Primitivity and repetition

E. Czeizler, L. Kari, S. Seki (2010)

AGT CAT

AGTCAT

AGTATG

$$u \odot^{\theta} v = \{uv, u\theta(v)\}$$

# Pseudocatenation

L. Kari and M. Kulkarni (2014)

AGGTACGTACCTGTACCTAGGTACAGGTAC

$$w = x_1 x_2 \cdots x_k, x_i \in \{x_1, \theta(x_1)\}$$

$$w \in u\{u, \theta(u)\}^*$$

$$w \in u^{\odot_k^\theta}$$

The **state complexity of a regular language** is the number of states in its minimal deterministic finite automaton.

The **state complexity of an operation** is the worst-case state complexity of the language resulting from the operation, as a function of the state complexity of the operands.

# Prior state complexity results

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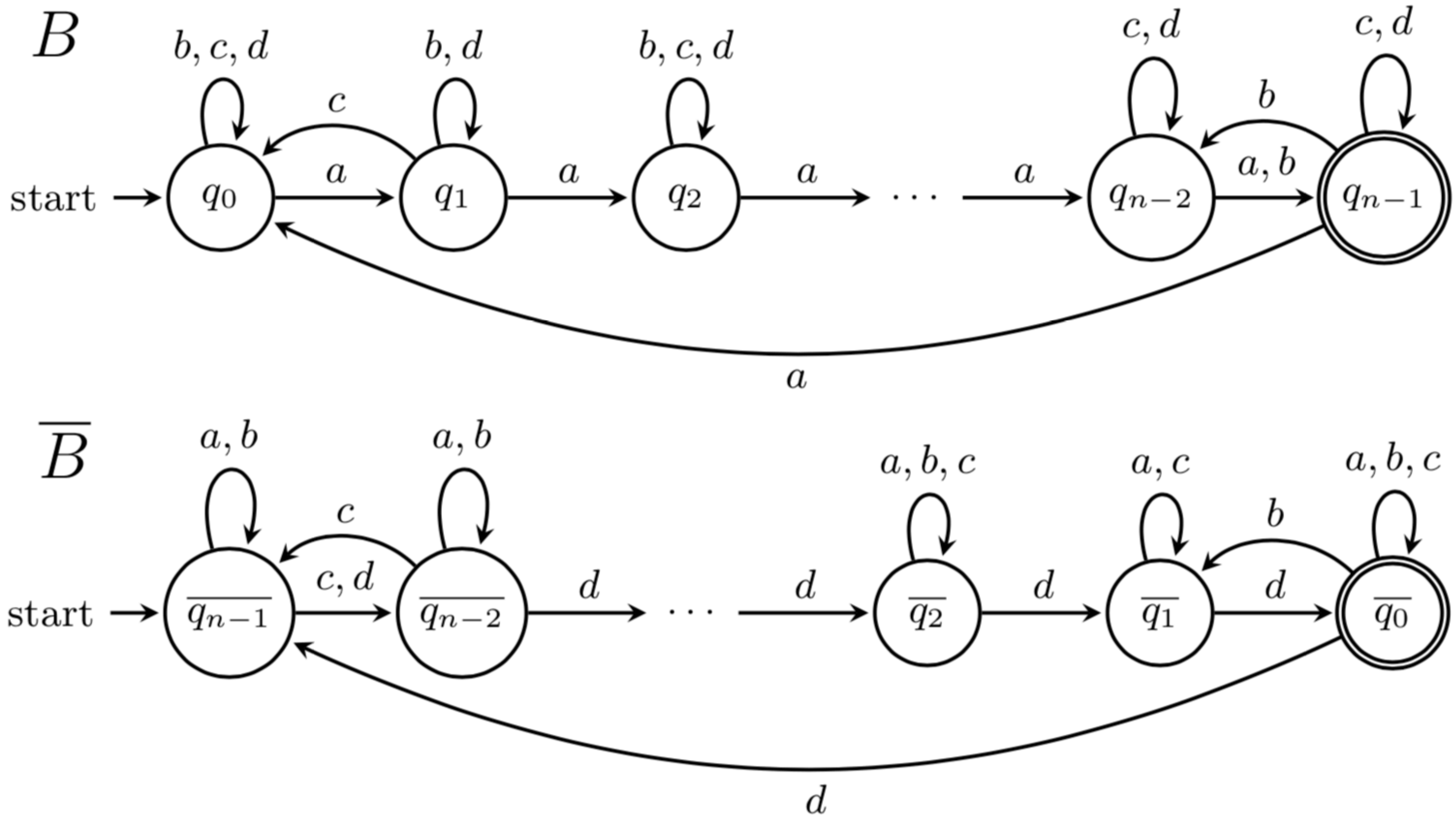
- Undecidability (A. Salomaa, K. Salomaa, S. Yu, 2013)
- Universal witnesses (J.A. Brzozowski, 2012)

# Prior state complexity results

- Hairpin-free languages (L. Kari et al. 2006)
- (Pseudo-)Inversion (D.-J. Cho et al. 2016)
- (Pseudo-)Duplication (D.-J. Cho et al. 2016)
- Overlap assembly (J.A. Brzozowski et al. 2018)

# Regular languages are closed under antimorphism

- Regular languages are closed under morphism
- Regular languages are closed under reversal
- Antimorphism is morphism composed with reversal



B
start
$q_0$
$q_1$
$q_2$
$q_{n-2}$
$q_{n-1}$
$b, c, d$
$b, d$
$b, c, d$
$c, d$
$c, d$
$c$
$a$
$a$
$a$
$a$
$b$
$a, b$
$a$
$\overline{B}$
start
$\overline{q_{n-1}}$
$\overline{q_{n-2}}$
$\overline{q_2}$
$\overline{q_1}$
$\overline{q_0}$
$a, b$
$a, b$
$a, b, c$
$a, c$
$a, b, c$
$c$
$c, d$
$d$
$d$
$d$
$b$
$d$
$d$

$$L_1 \odot^{\theta} L_2 = L_1(L_2 \cup \theta(L_2))$$

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$$L_m \cdot (L_n \cup L_p)$$

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$$L_m \cdot (L_n \cup L_p)$$

**B. Cui et al. 2011**

$$
\begin{aligned}
Q_C = & \{\langle q, P, \overline{R}\rangle \mid q \in Q_A - F_A, P \in 2^{Q_B} - \{\emptyset\}, \overline{R} \in 2^{\overline{Q_B}} - \{\emptyset\}\} \\
& \cup \{\langle q, \emptyset, \emptyset\rangle \mid q \in Q_A - F_A\} \\
& \cup \{\langle q, P \cup \{s_B\}, \overline{R} \cup \overline{F_B}\rangle \mid q \in F_A, P \in 2^{Q_B - \{s_B\}}, \overline{R} \in 2^{\overline{Q_B} - \overline{F_B}}\},
\end{aligned}
$$

**Machine A non-final**

$$
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\end{aligned}
$$

**Machine A first pass**

Machine A non-final

$$
\begin{aligned}
Q_C = & \{\langle q, P, \overline{R}\rangle \mid q \in Q_A - F_A, P \in 2^{Q_B} - \{\emptyset\}, \overline{R} \in 2^{\overline{Q_B}} - \{\emptyset\}\} \\
& \cup \{\langle q, \emptyset, \emptyset\rangle \mid q \in Q_A - F_A\} \\
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\end{aligned}
$$

Machine A first pass

Machine A final

$(m\text{-}k^A)(2^n\text{-}1)(2^n\text{-}1)$

$$
\begin{aligned}
Q_C = & \{\langle q, P, \overline{R}\rangle \mid q \in Q_A - F_A, P \in 2^{Q_B} - \{\emptyset\}, \overline{R} \in 2^{\overline{Q_B}} - \{\emptyset\}\} \\
& \cup \{\langle q, \emptyset, \emptyset\rangle \mid q \in Q_A - F_A\} \\
& \cup \{\langle q, P \cup \{s_B\}, \overline{R} \cup \overline{F_B}\rangle \mid q \in F_A, P \in 2^{Q_B - \{s_B\}}, \overline{R} \in 2^{\overline{Q_B} - \overline{F_B}}\},
\end{aligned}
$$

$m\text{-}k_A$

$k_A(2^{n\text{-}1})(2^{n\text{-}kB})$

$(m\text{-}k^A)(2^n\text{-}1)(2^n\text{-}1)$

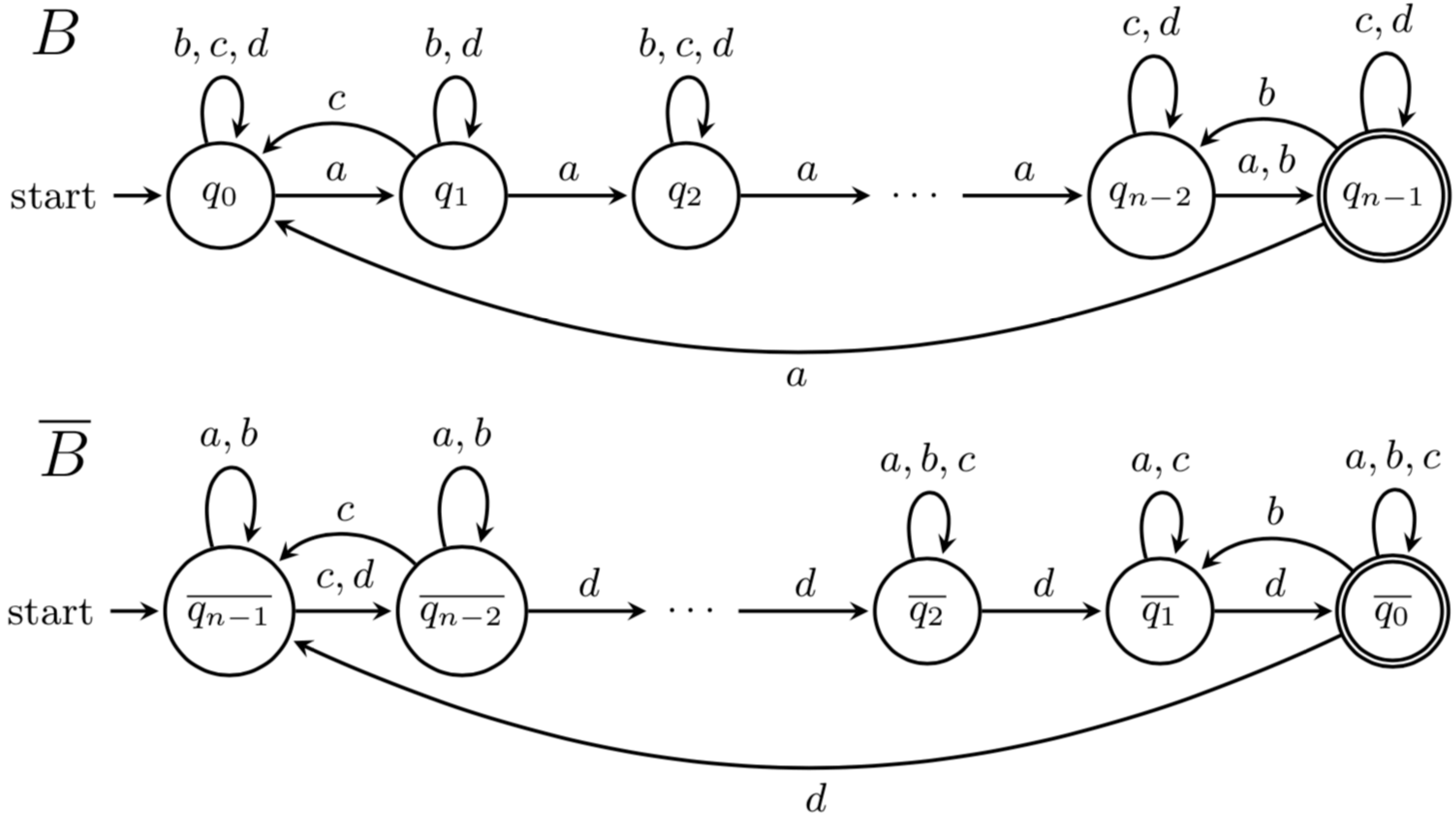
$$Q_C = \{\langle q, P, \overline{R}\rangle \mid q \in Q_A - F_A, P \in 2^{Q_B} - \{\emptyset\}, \overline{R} \in 2^{\overline{Q_B}} - \{\emptyset\}\}$$

$$\cup \{\langle q, \emptyset, \emptyset\rangle \mid q \in Q_A - F_A\}$$

$m\text{-}k_A$

$$\cup \{\langle q, P \cup \{s_B\}, \overline{R} \cup \overline{F_B}\rangle \mid q \in F_A, P \in 2^{Q_B - \{s_B\}}, \overline{R} \in 2^{\overline{Q_B} - \overline{F_B}}\},$$

$k_A(2^{n\text{-}1})(2^{n\text{-}kB}\text{-}1)+1$




# Lower bound

J.A. Brzozowski and D. Liu, 2013

| Operation | State complexity |
|---|---|
| $L_m \odot^{\theta} L_n$ | $(m-1)(2^{2n} - 2^{n+1} + 2) + 2^{2n-2} - 2^{n-1} + 1$ |
| $L_m L_n$ | $m2^n - 2^{n-1}$ |
| $L_m(L_n \cup L_p)$ | $(m-1)(2^{n+p} - 2^n - 2^p + 2) + 2^{n+p-2}$ |

| Operation | State complexity |
| --- | --- |
| $L_n^{\odot_2^\theta}$ | $(n-1)(2^{2n} - 2^{n+1} + 2) + 2^{2n-2} - 2^{n-1} + 1$ |
| $L_n^2$ | $n2^n - 2^{n-1}$ |

# Pseudosquare

(Square: N. Rampersad, 2006)

AGGTACGTACCTGTACCTAGGTACAGGTAC

$$w = x_1 x_2 \cdots x_k, x_i \in \{x_1, \theta(x_1)\}$$

**AGGTACGTACCTGTACCTAGGTACAGGTAC**

$$w = x_1 x_2 \cdots x_k, x_i \in \{x_1, \theta(x_1)\}$$

$$L^{\odot^{\theta}_{+}}, L = \{AG, AGT, TG, CGT, TT\}$$

**AGGTACGTACCTGTACCTAGGTACAGGTAC**

$$w = x_1 x_2 \cdots x_k, x_i \in \{x_1, \theta(x_1)\}$$

**AGAGTACTCGTAAACGCTAGT**

$$L^{\odot^{\theta}_{+}}, L = \{AG, AGT, TG, CGT, TT\}$$

$$L^{\odot^{\theta}_{+}} = L(L \cup \theta(L))^*$$

$$L^{\odot^{\theta}_{+}} = L(L \cup \theta(L))^*$$

$$(L_m \cup L_n)^*$$

$$L^{\odot^{\theta}_{+}} = L(L \cup \theta(L))^*$$

$$(L_m \cup L_n)^*$$

**A. Salomaa, K. Salomaa, S. Yu (2007)**

$$Q' = Q_1 \cup Q_2,$$
$$Q_1 = \{\langle P, \overline{R}\rangle \mid \emptyset \neq P \subseteq Q - F, \overline{R} \subseteq \overline{Q} - \{\overline{s}\}\},$$
$$Q_2 = \{\langle P \cup \{s\}, \overline{R} \cup \overline{F}\rangle \subseteq Q \times \overline{Q} \mid (P \cup \overline{R}) \cap (F \cup \{\overline{s}\}) \neq \emptyset\}.$$

**Non-final states**

$$Q' = Q_1 \cup Q_2,$$
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**Including final states**

$(2^{n\text{-}k}\text{-}1)(2^{n\text{-}1})$

$$Q' = Q_1 \cup Q_2,$$

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$(2^{n\text{-}1})(2^{n\text{-}k})$

$$(2^{n-k}-1)(2^{n-1})$$

$$Q' = Q_1 \cup Q_2,$$
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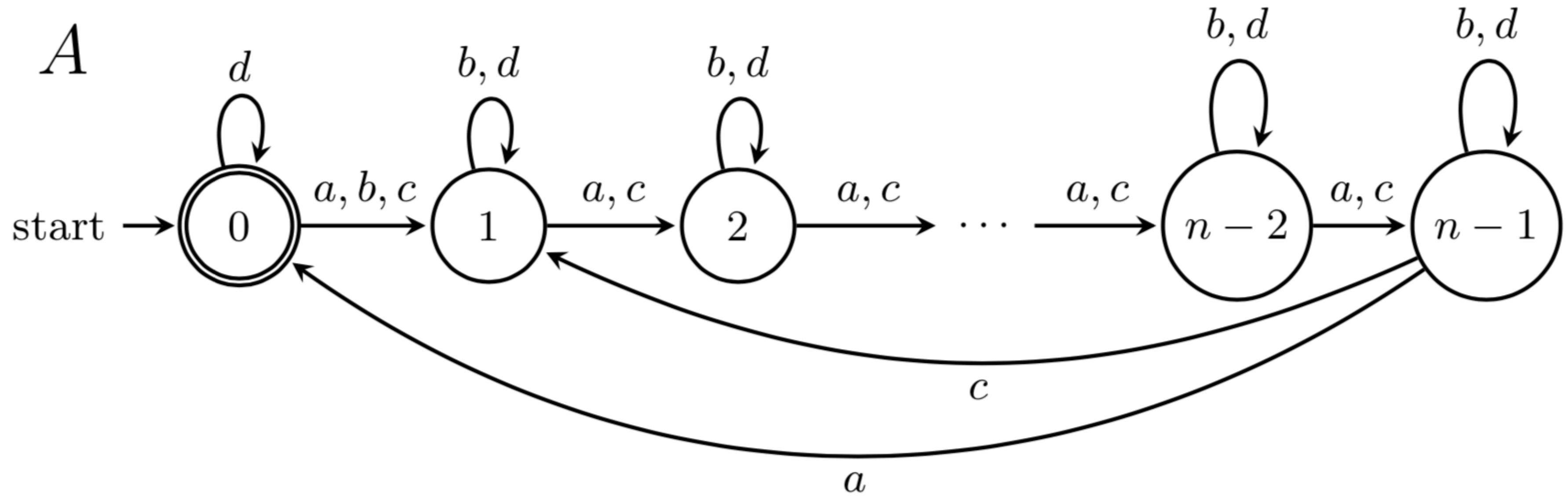
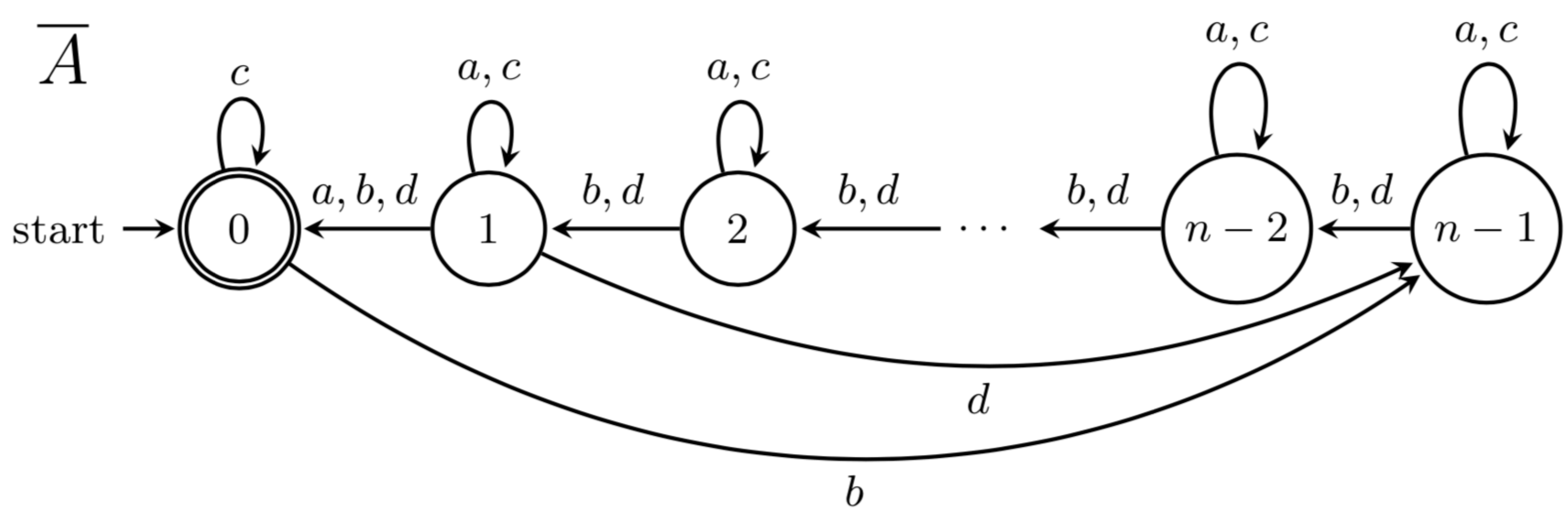
$$(2^{n-1})(2^{n-k})$$

**This changes further depending on whether or not $s \in F$.**


$A$
start
0
1
2
$n-2$
$n-1$
$\cdots$
$d$
$b, d$
$b, d$
$b, d$
$b, d$
$a, b, c$
$a, c$
$a, c$
$a, c$
$a, c$
$c$
$a$


$\overline{A}$
start
0
1
2
$n-2$
$n-1$
$\cdots$
$c$
$a, c$
$a, c$
$a, c$
$a, c$
$a, b, d$
$b, d$
$b, d$
$b, d$
$b, d$
$d$
$b$

# Lower bound

| Operation | State complexity |
| --- | --- |
| $L_n^{\odot_+^\theta}$ | $2^{2n-1} - 2^n + 1$ |
| $L_n^*$ | $2^{n-1} + 2^{n-2}$ |
| $(L_m \cup L_n)^*$ | $2^{m+n-1} - 2^{m-1} - 2^{n-1} + 1$ |

# Future work

$k^{th}$ pseudopower and pseudocube,
finite languages, universal witnesses

# Finite languages

- State complexity of operations on finite languages is often much lower (C. Câmpeanu et al. 1999)
- Starting with a finite set and generating sets of words makes sense from the biological perspective

# Universal witnesses

- Universal witnesses used in pseudocatenation are not the ones for combinations of operations with reversal!
- Lower bound witness for iterated pseudocatenation is not a universal witness as defined by Brzozowski
- Are there universal witnesses for combinations of operations involving antimorphisms?

# $k^{th}$ pseudopower

- State complexity of power and cube (M. Domaratzki and A. Okhotin, 2009)
- State complexity of square and power for unary languages (N. Rampersad, 2006)
- State complexity of k-catenation (Ésik et al. 2009, Caron et al. 2016)

# Thank you